

VIRTUAL CROSSINGS AND A FILTRATION OF THE TRIPLY GRADED HOMOLOGY OF A LINK DIAGRAM

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ABSTRACT. A filtration of Soergel bimodules by virtual crossing bimodules extends to Rouquier's complexes associated with braid words. We show that these complexes are invariant up to filtered homotopy with respect to the second Reidemeister move, and the filtration of the triply graded link diagram homology, constructed by Khovanov through the application of the Hochschild homology, is invariant under Markov moves. We also prove that the homotopy equivalence of the complexes of braid words related by the third Reidemeister move violates filtration by at most two units.

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1. INTRODUCTION

In [KR07] the virtual crossing categorification was introduced as a necessary tool in categorifying the $SO(2N)$ Kauffman polynomial. In the same paper the categorification of $SU(N)$ and triply graded link homology were also extended to virtual links. The results of that paper inspired an attempt by Emmanuel Wagner [Wag] to introduce the third grading to the $SU(N)$ link homology. Unfortunately, his approach contained substantial errors and the paper [Wag] was subsequently withdrawn.

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In this paper we attempt, in the spirit of Wagner's ideas, to use virtual crossing complexes of [KR07] in order to introduce a *filtration* in the triply graded link homology. A similar filtration can be introduced in the $SU(N)$ link homology.

Recall that two chain complexes (A, d_A) and (B, d_B) of objects of an additive category \mathcal{C} are homotopy equivalent if there exists a diagram

$$h_A \curvearrowright A \begin{array}{c} \xrightarrow{f_{AB}} \\ \xleftarrow{f_{BA}} \end{array} B \curvearrowright h_B \quad (1.1)$$

whose morphisms satisfy the relations

$$d_B f_{AB} - f_{AB} d_A = 0, \quad d_A f_{BA} - f_{BA} d_B = 0, \quad (1.2)$$

$$f_{BA} f_{AB} - \mathbb{1}_A = [d_A, h_A], \quad f_{AB} f_{BA} - \mathbb{1}_B = [d_B, h_B]. \quad (1.3)$$

We call f_{AB}, f_{BA} *homotopy equivalences* and we call h_A, h_B *homotopies*.

Suppose that the complexes (A, d_A) and (B, d_B) are filtered, that is, the chain objects of A, B are filtered and the differentials d_A, d_B are filtered morphisms. We say that (A, d_A) and (B, d_B) are *filtered homotopy equivalent* if all morphisms of the diagram (1.1) are filtered.

Let \mathbf{s} be the filtration shift functor: for a filtered object A we have $F^{(i)}(\mathbf{s}A) = F^{(i-1)}A$. We say that a morphism between two filtered objects $f: A \rightarrow B$ *violates* filtration by k units if its shifted version $f: M \rightarrow \mathbf{s}^k N$ is filtered.

The definition of the triply graded homology requires a presentation of a link L as a circular closure of a braid word β , which by definition is a finite sequence of positive and negative elementary braids σ_i and σ_i^{-1} , $1 \leq i \leq n-1$, where n is the number of strands in β . Following Soergel [Soe92, Soe] and Rouquier [Rou], to elementary braids σ_i and σ_i^{-1} we associate complexes $[\![\sigma_i]\!]$ and $[\![\sigma_i^{-1}]\!]$ of Soergel bimodules. However this time each Soergel bimodule in these complexes is assigned a descending filtration and the complexes are also filtered, that is, the differentials are filtered homomorphisms. Remarkably, this filtration is related to virtual crossings: an associated graded of a Soergel bimodule is a sum of bimodules corresponding to purely virtual braids.

For an n -strand braid word β we introduce two sets of n variables: $\mathbf{x} = x_1, \dots, x_n$ and similarly \mathbf{y} . A filtered Soergel bimodule ${}_y M_{\mathbf{x}}$ is an object in the derived category $\mathbf{D}(\mathbb{Q}[\mathbf{x}, \mathbf{y}] - \mathbf{fi}, \mathbf{g})$ of \mathbb{Z} -graded (q -grading), filtered $\mathbb{Q}[\mathbf{x}, \mathbf{y}]$ -modules. Then a complex of bimodules ${}_y [\![\beta]\!]_{\mathbf{x}}$ associated with β is an object in the category $\mathbf{Ch}(\mathbf{D}(\mathbb{Q}[\mathbf{x}, \mathbf{y}] - \mathbf{fi}, \mathbf{g}))$ of chain complexes over an additive category $\mathbf{D}(\mathbb{Q}[\mathbf{x}, \mathbf{y}] - \mathbf{fi}, \mathbf{g})$. As usual, the complex ${}_y [\![\beta]\!]_{\mathbf{x}}$ is determined by the choice of complexes $[\![\sigma_i]\!]$ and $[\![\sigma_i^{-1}]\!]$ and by the composition rule: a complex associated to the product of braid words is the tensor product of their complexes over the intermediate variables: ${}_z [\![\beta_2 \beta_1]\!]_{\mathbf{x}} = {}_z [\![\beta_2]\!]_{\mathbf{y}} \otimes_{\mathbb{Q}[\mathbf{y}]} {}_y [\![\beta_1]\!]_{\mathbf{x}}$.

Note that an object of $\mathbf{Ch}(\mathbf{D}(\mathbb{Q}[\mathbf{x}, \mathbf{y}] - \mathbf{fi}, \mathbf{g}))$ is a complex of complexes and, as such, it has two differentials: an inner differential corresponding to $\mathbf{D}(-)$ and an outer one corresponding to $\mathbf{Ch}(-)$. We denote the corresponding homologies as H_{in} and H_{out} respectively.

Let L_β be the link diagram which is a circular closure of a braid word β . We define its associated complex $\llbracket L_\beta \rrbracket$ as the result of replacing the bimodules of ${}_y\llbracket \beta \rrbracket_x$ by their derived tensor products with the ‘diagonal’ bimodule corresponding to the identity braid $\mathbb{1}_n$ (see eq. (6.1)). The complex $\llbracket L_\beta \rrbracket$ is an object in the category of complexes over the homotopy category of complexes of filtered, graded vector spaces: $\mathbf{Ch}(\mathbf{K}(\mathbb{Q} - \mathbf{fi}, \mathbf{g}))$. The homology of $\llbracket L_\beta \rrbracket$ with respect to the inner differential is the Hochschild homology of ${}_y\llbracket \beta \rrbracket_x$. The subsequent application of the outer homology produces the filtered version of the triply graded homology of L_β : $H^{(3)}(\llbracket L_\beta \rrbracket) = H_{\text{out}}(H_{\text{in}}(\llbracket L_\beta \rrbracket))$.

A braid is determined by a braid word up to the second and third Reidemeister moves:

$$\sigma_i \sigma_i^{-1} = \mathbb{1}_n \quad (\text{R2a}), \quad \sigma_i^{-1} \sigma_i = \mathbb{1}_n \quad (\text{R2b}), \quad \sigma_i^{-1} \sigma_{i+1}^{-1} \sigma_i^{-1} = \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1} \quad (\text{R3}),$$

where $\mathbb{1}_n$ is the unit braid with n strands. We will show that if two braid words are related by a second Reidemeister move, then their complexes are filtered homotopy equivalent (Theorem 4.1). If braid words β_1 and β_2 are related by a third Reidemeister move, then we can *not* prove the filtered homotopy equivalence of their complexes. However, we prove that the homotopy equivalences established by Rouquier

$$\llbracket \beta_1 \rrbracket \begin{array}{c} \xrightarrow{f_{12}} \\ \xleftarrow{f_{21}} \end{array} \llbracket \beta_2 \rrbracket$$

violate filtration by at most two units (Theorem 5.1).

An oriented framed link is determined by a braid up to the first and second Markov moves:

$$L_{\beta_1 \beta_2} = L_{\beta_2 \beta_1} \quad (\text{M1}), \quad L_{(\sigma_1 \sqcup \mathbb{1}_n)(\mathbb{1}_1 \sqcup \beta)} = L_\beta \quad (\text{M2a}), \quad L_{(\sigma_1^{-1} \sqcup \mathbb{1}_n)(\mathbb{1}_1 \sqcup \beta)} = L_\beta \quad (\text{M2b}),$$

We will show that if two braid words β_1 and β_2 are related by M1 or M2a, then the complexes of their closures are filtered homotopy equivalent: $\llbracket L_{\beta_1} \rrbracket \sim \llbracket L_{\beta_2} \rrbracket$, whereas if they are related by M2b, then the relation between their complexes is weaker: their inner homologies are filtered homotopy equivalent (Remark 6.1 and Theorem 6.3). As a consequence, if β_1 and β_2 are related by any sequence of Markov moves, then their inner homologies are filtered homotopy equivalent: $H_{\text{in}}(\llbracket L_{\beta_1} \rrbracket) \sim H_{\text{in}}(\llbracket L_{\beta_2} \rrbracket)$ and their triply graded filtered homologies are isomorphic: $H^{(3)}(\llbracket L_{\beta_1} \rrbracket) \cong H^{(3)}(\llbracket L_{\beta_2} \rrbracket)$.

In Section 7 we find a simple presentation of the filtered complex of a two-strand braid word (Theorem 7.1) and compute the filtered homology of two-strand torus knots and links presented as their closures (Theorem 7.3).

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2. FILTERED ROUQUIER COMPLEX OF A BRAID WORD

2.1. Overview. A convenient description of algebraic constructions related to all versions of the triply graded link homology uses various versions of a monoidal 2-category of points \mathbf{Pt} . An object in this category is a finite number of ordered points: (n) is an object of n points. A monoidal product of two objects is their ordered disjoint union $(n_1) \sqcup (n_2)$. In this paper we assume that the only morphisms between objects are endomorphisms. An endomorphism of an object is a 1-dimensional cobordism of some sort: a braid, a braid word, a graph-braid, a virtual braid, *etc.* The endomorphisms are composed as cobordisms and their monoidal product is again a disjoint union. The set of endomorphisms $\text{End}(n)$ usually has a category structure: a morphism between two 1-dimensional cobordisms is a 2-dimensional cobordism.

We use a shortcut notation $\mathbf{x} = x_1, \dots, x_n$ for a list of indexed variables, with $|\mathbf{x}| = n$ denoting the number of variables in a list.

A link homology construction involves a categorification functor $\llbracket - \rrbracket$ from the 2-category \mathbf{Pt} to a monoidal algebraic 2-category. To an object (n) the functor associates an algebra $\mathbb{Q}[\mathbf{x}]$, $|\mathbf{x}| = n$, the variables $\mathbf{x} = x_1, \dots, x_n$ being associated with the point of (n) . Then the functor $\llbracket - \rrbracket$ maps the category of endomorphisms $\text{End}(n)$ to an appropriate category of (graded and filtered) bimodules over $\mathbb{Q}[\mathbf{x}]$, that is, to a derived category of $\mathbb{Q}[\mathbf{x}, \mathbf{y}]$ -modules or to a category of chain complexes over it. We refer to these categories as *target categories*.

To a composition of endomorphisms in $\text{End}(n)$ the functor $\llbracket - \rrbracket$ associates the composition of bimodules defined as the (derived) tensor product over the intermediate variables, whose action is then forgotten (we call such variables *dummy*), while to the disjoint union of endomorphisms one associates the tensor product of bimodules over \mathbb{Q} . Finally, to a morphism between two endomorphisms in $\text{End}(n)$ the functor $\llbracket - \rrbracket$ associates a homomorphism of corresponding bimodules or a chain map between their complexes.

The language of the category \mathbf{Pt} allows a simple definition of the endomorphisms $\text{End}(n)$: they are generated by the identity endomorphism $\mathbb{1}_{(1)} \in \text{End}(1)$, which we denote graphically as $\mathbb{1}_{(1)} = |$, and by a finite number of ‘simple’ endomorphisms $\epsilon_i \in \text{End}(2)$ through composition and monoidal product modulo certain relations such as the symmetry group relations or the braid group relations. The functor $\llbracket - \rrbracket$ is defined by its action on the generators: $\llbracket \mathbb{1}_{(1)} \rrbracket$

and $\llbracket \epsilon_i \rrbracket$, and the main challenge is to verify that, thus defined, $\llbracket - \rrbracket$ respects the relations between them.

In this paper we consider three types of endomorphisms in $\text{End}(n)$. The endomorphisms of the first type form the group of virtual braids or, equivalently, the symmetric group S_n . Apart from the unit element $|$, virtual braids are generated by the elementary virtual braid (*i.e.* permutation) $s_{12} \in \text{End}(2)$, presented graphically as $s_{12} = \times$, modulo the relations of S_n (*i.e.* the purely virtual second and third Reidemeister moves). A generator s_i of S_n has a graphical presentation as a virtual braid:

$$s_i = \underbrace{|\cdots|}_{i-1} \times \underbrace{|\cdots|}_{n-i-1}.$$

The second type of endomorphisms in $\text{End}(n)$ are *braid-graphs*. These endomorphisms are generated freely by $|$ and by two blob generators $\bullet \times, \times \bullet \in \text{End}(2)$. More precisely, the monoid of n -strand braid-graphs is freely generated by two elements:

$$b_i^+ = \underbrace{|\cdots|}_{i-1} \times \bullet \underbrace{|\cdots|}_{n-i-1}, \quad b_i^- = \underbrace{|\cdots|}_{i-1} \bullet \times \underbrace{|\cdots|}_{n-i-1}. \quad (2.1)$$

One can mix braid-graphs with virtual braids modulo the relations

$$\begin{array}{c} \diagup \diagdown \\ \bullet \end{array} \cong \begin{array}{c} \diagup \diagdown \\ \bullet \end{array} \cong \times \bullet, \quad (2.2)$$

which, in view of the second virtual Reidemeister move, imply

$$\begin{array}{c} \diagup \diagdown \\ \bullet \end{array} \cong \begin{array}{c} \diagup \diagdown \\ \bullet \end{array} \cong \times \bullet. \quad (2.3)$$

The third type of endomorphisms in $\text{End}(n)$ are braid words which form a monoid. These monoids are generated freely by $|$ and by two elementary braids $\succ, \prec \in \text{End}(2)$. In other words, the monoid of n -strand braid words is generated by the elements

$$\sigma_i = \underbrace{|\cdots|}_{i-1} \succ \underbrace{|\cdots|}_{n-i-1}, \quad \sigma_i^- = \underbrace{|\cdots|}_{i-1} \prec \underbrace{|\cdots|}_{n-i-1}.$$

One could pass from the braid word monoid to the braid group by imposing the second and third Reidemeister move relations, but we do not do it, because in our filtered context we can not prove the invariance of the functor $\llbracket - \rrbracket$ under the third Reidemeister move.

As usual, braids can be mixed with virtual braids and braid-graphs, but we do not study possible relations, except the following two:

$$\begin{array}{c} \diagup \diagdown \\ \diagup \diagdown \end{array} \cong \succ, \quad \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \cong \prec \quad (2.4)$$

2.2. Target categories. Our target categories are the usual ones with an extra filtration. The first target category is the derived category of filtered, \mathbb{Z} -graded $\mathbb{Q}[\mathbf{x}, \mathbf{y}]$ -modules: $\mathbf{D}(\mathbb{Q}[\mathbf{x}, \mathbf{y}] - \mathbf{fi}, \mathbf{g})$. The \mathbb{Z} -grading and its associated q -degree comes from assigning degrees to variables: $\deg_q x_i = \deg_q y_i = 2$; this grading is always present and we drop it from the category notations. Apart from filtration and q -grading, the category $\mathbf{D}(\mathbb{Q}[\mathbf{x}, \mathbf{y}] - \mathbf{fi}, \mathbf{g})$ has a homological \mathbb{Z} -grading with a -degree denoted as \deg_a . It determines all sign factors associated with homological algebra. The category $\mathbf{D}(\mathbb{Q}[\mathbf{x}, \mathbf{y}] - \mathbf{fi}, \mathbf{g})$ is endowed with three degree shift endofunctors related to the q -grading, a -grading and filtration: \mathbf{q} , \mathbf{a} and \mathbf{s} .

A filtered $\mathbb{Q}[\mathbf{x}, \mathbf{y}]$ -module is called *filtered semi-free* if it is split-free as a $\mathbb{Q}[\mathbf{x}]$ -module and as a $\mathbb{Q}[\mathbf{y}]$ -module. All bimodules in this paper are filtered semi-free, therefore one can use an ordinary tensor product over the intermediate variables rather than a derived one when composing bimodules by taking their tensor product over the intermediate variables.

The category $\mathbf{D}(\mathbb{Q}[\mathbf{x}, \mathbf{y}] - \mathbf{fi}, \mathbf{g})$ is additive, and our second target category is the category of bounded chain complexes over it: $\mathbf{Ch}(\mathbf{D}(\mathbb{Q}[\mathbf{x}, \mathbf{y}] - \mathbf{fi}, \mathbf{g}))$. An object of $\mathbf{Ch}(\mathbf{D}(\mathbb{Q}[\mathbf{x}, \mathbf{y}] - \mathbf{fi}, \mathbf{g}))$ is a ‘complex of complexes’. This category has yet another shift endofunctor \mathbf{t} whose associated grading reflects the homological degree within the complexes of $\mathbf{Ch}(-)$. However \deg_t is not a homological degree, the latter being \deg_a , and, in fact, \deg_t may take semi-integer values. In order to simplify our formulas, we introduce combined shift endofunctors:

$$\mathbf{a}_q = \mathbf{a}\mathbf{q}^2, \quad \mathbf{s}_q = \mathbf{s}\mathbf{q}^2, \quad \mathbf{t}_a = \mathbf{t}\mathbf{a}, \quad \mathbf{t}_s = (\mathbf{t}\mathbf{s})^{1/2}.$$

In order to distinguish the inner and outer complexes within a complex of complexes, we put the inner complexes in boxes, while the outer complexes are in square brackets. Here are two examples of this notation:

$$\left[\mathbf{t}_a \left[\mathbf{a} M_1 \xrightarrow{g_M} M_2 \right] \xrightarrow{f} \left[\mathbf{a} N_1 \xrightarrow{g_N} N_2 \right] \right] = \left[\mathbf{t}_a \left[\begin{array}{ccc} \mathbf{a} M_1 & \xrightarrow{f_1} & \mathbf{a} N_1 \\ \downarrow g_M & & \downarrow g_N \\ M_2 & \xrightarrow{f_2} & N_2 \end{array} \right] \right],$$

where M_1 , M_2 , N_1 and N_2 are $\mathbb{Q}[\mathbf{x}, \mathbf{y}]$ -modules, while g_M , g_N , f_1 and f_2 are homomorphisms between them. Note that the *r.h.s.* presents the chain morphism f explicitly in terms of its components f_1 and f_2 .

2.3. Filtered modules and their resolutions. Let $M = M_0 \supset \cdots \supset M_m$ be a filtered $\mathbb{Q}[\mathbf{x}, \mathbf{y}]$ -module representing an object in the derived category $\mathbf{D}(\mathbb{Q}[\mathbf{x}, \mathbf{y}] - \mathbf{fi}, \mathbf{g})$. Denote $N_i = M_i/M_{i+1}$ the associated graded submodules. Then a filtered projective resolution $P(M)$ of M can be constructed as a multiple cone, that is, a sum of projective resolutions

$P(N_i)$ with added differentials $f_{ij}: N_i \rightarrow N_j$, $i < j$, which are not necessarily closed, except when $j = i + 1$. For example, if the filtration depth m is three, then

$$P(M) \sim \boxed{\begin{array}{c} \begin{array}{ccccccc} & & & f_{03} & & & \\ & & \nearrow f_{02} & & \searrow & & \\ P(N_0) & \xrightarrow{f_{01}} & sP(N_1) & \xrightarrow{f_{12}} & s^2P(N_2) & \xrightarrow{f_{23}} & s^3P(N_3) \\ & & \searrow & & \nearrow f_{13} & & \\ & & & f_{13} & & & \end{array} \end{array}}.$$

We call this picture a filtration diagram of M .

Let the $\mathbb{Q}[\mathbf{x}, \mathbf{y}]$ -module M have a filtration of depth $m = 1$: $M = M_0 \supset M_1$, then $N_0 = M_0/M_1$ and these modules form an exact sequence

$$0 \rightarrow M_1 \rightarrow M \rightarrow N_0 \rightarrow 0. \quad (2.5)$$

The latter turns into an exact triangle of filtered resolutions:

$$sP(M_1) \xrightarrow{\mathbb{1}} \boxed{\begin{array}{c} P(N_0) \\ \downarrow f \\ sP(M_1) \end{array}} \xrightarrow{\mathbb{1}} P(N_0) \xrightarrow{f} P(M_1)$$

where the chain map f represents the first extension determined by the exact sequence (2.5).

2.4. Symmetric group and special bimodules. The simplest choice of $\text{End}(n)$ is the symmetry group: $\text{End}(n) = S_n$. From the topological point of view S_n is the group of purely virtual braids (a categorification of the whole virtual braid group is discussed in [Thi11]). From the **Pt** point of view, the groups S_n are generated by the identity 1-strand braid $\mathbb{1}_{(1)} \in \text{End}(1)$ and by the elementary permutation $s_{12} \in \text{End}(2)$ which generates S_2 . These generators have a graphic presentation: $\mathbb{1}_{(1)} = |$, $s_{12} = \times$. Note that we do not put a circle around a virtual crossing.

The target category for $\text{End}(n) = S_n$ is just $\mathbf{D}(\mathbb{Q}[\mathbf{x}, \mathbf{y}] - \mathbf{fi}, \mathbf{g})$ and we set

$$[\mathbb{1}] = \mathbb{Q}[x, y]/(y - x), \quad [\times] = \mathbb{Q}[\mathbf{x}, \mathbf{y}]/(y_2 - x_1, y_1 - x_2), \quad |\mathbf{x}| = |\mathbf{y}| = 2.$$

This choice dictates the bimodule to be associated with a general permutation $s \in S_n$:

$$[s] = \mathbb{Q}[\mathbf{x}, \mathbf{y}]/(y_{s(1)} - x_1, \dots, y_{s(n)} - x_n). \quad (2.6)$$

All bimodules $\llbracket s \rrbracket$ are endowed with the trivial filtration: $F_0 \llbracket s \rrbracket = \llbracket s \rrbracket$ and $F_i \llbracket s \rrbracket = 0$ if $i > 0$. It is easy to see that the bimodules (2.6) satisfy S_n relations, that is,

$${}_z \llbracket s' \rrbracket_y \otimes_{\mathbb{Q}[y]} {}_y \llbracket s \rrbracket_x \cong {}_z \llbracket s' s \rrbracket_x \quad (2.7)$$

for any $s, s' \in S_n$.

Remark 2.1. *The tensor product functors ${}_y \llbracket s \rrbracket_x \otimes_{\mathbb{Q}[x]} -$ and $- \otimes_{\mathbb{Q}[y]} {}_y \llbracket s \rrbracket_x$ act by permuting the corresponding variables.*

The bimodule (2.6) can be presented as a tensor product of $\llbracket \cdot \rrbracket$ -like bimodules over \mathbb{Q} : $\llbracket s \rrbracket \cong \bigotimes_{i=1}^n {}_{y_{s(i)}} \llbracket \cdot \rrbracket_{x_i}$. Denote the *canonical* Koszul resolution of the ‘diagonal’ bimodule $\llbracket \cdot \rrbracket$ as ${}_y \Delta_x$:

$${}_y \llbracket \cdot \rrbracket_x^{\text{cn}} = {}_y \Delta_x = \boxed{{}_q \mathbf{a}_q \mathbb{Q}[x, y] \xrightarrow{y-x} \mathbb{Q}[x, y]}. \quad (2.8)$$

The index ‘cn’ means ‘canonical’. The canonical Koszul resolution of the bimodule (2.6) is defined as the tensor product of resolutions (2.8):

$$\llbracket s \rrbracket^{\text{cn}} = \bigotimes_{i=1}^n {}_{y_{s(i)}} \Delta_{x_i}. \quad (2.9)$$

Denote

$$x_+ = x_1 + x_2, \quad x_- = x_2 - x_1, \quad y_+ = y_1 + y_2, \quad y_- = y_2 - y_1. \quad (2.10)$$

Then the canonical Koszul resolutions of $\llbracket \cdot \rrbracket \langle \cdot \rrbracket$ and $\llbracket \times \rrbracket$ split into tensor products of complexes of $\mathbb{Q}[x_+, y_+]$ -modules and complexes of $\mathbb{Q}[x_-, y_-]$ -modules:

$$\llbracket \cdot \rrbracket \langle \cdot \rrbracket^{\text{cn}} = {}_{y_+} \Delta_{x_+} \otimes \llbracket \cdot \rrbracket \langle \cdot \rrbracket^-, \quad \llbracket \times \rrbracket^{\text{cn}} = {}_{y_+} \Delta_{x_+} \otimes \llbracket \times \rrbracket^-, \quad (2.11)$$

where

$$\llbracket \cdot \rrbracket \langle \cdot \rrbracket^- = {}_{y_-} \Delta_{x_-}, \quad \llbracket \times \rrbracket^- = {}_{y_-} \Delta_{-x_-}. \quad (2.12)$$

Note that ${}_{y_+} \Delta_{x_+}$ is a common factor in these products. We introduce special morphisms $\phi_{\parallel} \in \text{Ext}^1(\llbracket \cdot \rrbracket \langle \cdot \rrbracket, \llbracket \times \rrbracket)$ and $\phi_{\times} \in \text{Ext}^1(\llbracket \times \rrbracket, \llbracket \cdot \rrbracket \langle \cdot \rrbracket)$: they act as identity on the common factor ${}_{y_+} \Delta_{x_+}$, while acting on the second factors as follows:

$$\begin{array}{ccc} \llbracket \cdot \rrbracket \langle \cdot \rrbracket^- \cong \boxed{{}_q \mathbf{a}_q \mathbb{Q}[x_-, y_-] \xrightarrow{y_- - x_-} \mathbb{Q}[x_-, y_-]} & & \llbracket \times \rrbracket^- \cong \boxed{{}_q \mathbf{a}_q \mathbb{Q}[x_-, y_-] \xrightarrow{y_- + x_-} \mathbb{Q}[x_-, y_-]} \\ \downarrow \phi_{\parallel} & \searrow 1 & \downarrow \phi_{\times} \\ \llbracket \times \rrbracket^- \cong \boxed{{}_q \mathbf{a}_q \mathbb{Q}[x_-, y_-] \xrightarrow{y_- + x_-} \mathbb{Q}[x_-, y_-]} & & \llbracket \cdot \rrbracket \langle \cdot \rrbracket^- \cong \boxed{{}_q \mathbf{a}_q \mathbb{Q}[x_-, y_-] \xrightarrow{y_- - x_-} \mathbb{Q}[x_-, y_-]} \end{array} \quad (2.13)$$

Obviously, $\deg_a \phi_{\parallel} = \deg_a \phi_{\times} = -1$, $\deg_q \phi_{\parallel} = \deg_q \phi_{\times} = -2$.

We refer to ϕ_{\parallel} and ϕ_{\times} as virtual saddle morphisms, because they may be considered as the result of applying the categorification functor to the virtual saddle cobordism connecting the virtual braids $\rangle \langle$ and \times .

For any permutation $s \in S_n$ and any transposition s_{ij} we define a virtual saddle morphism

$$\phi_{ij}^{(s)} \in \text{Ext}^1([\![s]\!], [\![ss_{ij}]\!]), \quad (2.14)$$

where $s, s_{ij} \in S_n$ and s_{ij} is a transposition. The bimodules $[\![s]\!]$ and $[\![ss_{ij}]\!]$ have a common factor $M_{\text{cm}} = \bigotimes_{k \neq i, j} y_{s(k)} [\![\]\!]_{x_k}$:

$$y[\![s]\!]_x = M_{\text{cm}} \otimes_{y_{s(i)}, y_{s(j)}} [\![\]\!]_{x_i, x_j}, \quad y[\![ss_{ij}]\!]_x = M_{\text{cm}} \otimes_{y_{s(i)}, y_{s(j)}} [\![\times]\!]_{x_i, x_j}$$

and $\phi_{ij}^{(s)}$ acts as identity on M_{cm} and as ϕ_{\parallel} on the remaining 2-strand factors.

In view of Remark 2.1, the tensor product with other permutation bimodules transforms the virtual saddle morphism by relabeling:

$$\begin{array}{ccc} w[\![s'']\!]_z \otimes_{Q[z]} z[\![s]\!]_y \otimes_{Q[y]} y[\![s']\!]_x & \xrightarrow{1 \otimes \phi_{ij}^{(s)} \otimes 1} & w[\![s'']\!]_z \otimes_{Q[z]} z[\![ss_{ij}]\!]_y \otimes_{Q[y]} y[\![s']\!]_x \\ \downarrow \cong & & \downarrow \cong \\ w[\![s''ss']\!]_x & \xrightarrow{\phi_{(s')^{-1}(i), (s')^{-1}(j)}^{(s''ss')}} & w[\![s''ss_{ij}s']\!]_x \end{array} \quad (2.15)$$

2.5. Filtered Soergel bimodules.

2.5.1. Filtration. The second step in building up $\text{End}(n)$ is the addition of two blob generators $\blacklozenge, \blacktriangleright \in \text{End}(2)$. The corresponding filtered bimodules are defined by endowing the standard Soergel blob bimodule

$$M_{\boxtimes} = \mathbb{Q}[\mathbf{x}, \mathbf{y}] / ((y_1 + y_2) - (x_1 + x_2), y_1 y_2 - x_1 x_2) = \mathbb{Q}[\mathbf{x}, \mathbf{y}] / (y_+ - x_+, (y_- + x_-)(y_- - x_-))$$

with a filtration related to the following exact sequences:

$$[\![\blacklozenge]\!]: \mathbf{q}^2[\![\times]\!] \xrightarrow{y_- - x_-} M_{\boxtimes} \xrightarrow{1} [\![\]\!] \langle \!| \!|, \quad [\![\blacktriangleright]\!]: \mathbf{q}^2[\![\]\!] \langle \!| \!| \xrightarrow{y_- + x_-} M_{\boxtimes} \xrightarrow{1} [\![\times]\!]. \quad (2.16)$$

In other words,

$$F_0[\![\blacklozenge]\!] = [\![\blacklozenge]\!], \quad F_1[\![\blacklozenge]\!] = \mathbf{q}^2[\![\times]\!], \quad F_0[\![\blacktriangleright]\!] = [\![\blacktriangleright]\!], \quad F_1[\![\blacktriangleright]\!] = \mathbf{q}^2[\![\]\!] \langle \!| \!|.$$

It is easy to see that the exact sequences (2.16) correspond to the extensions ϕ_{\parallel} and ϕ_{\times} defined by the diagrams (2.13). Canonical filtered resolutions of the bimodules $[\![\blacklozenge]\!]$ and

$\llbracket \bowtie \rrbracket$ are defined as cones:

$$\llbracket \bullet \bowtie \rrbracket^{\text{cn}} = \boxed{\llbracket \rangle \langle \rrbracket^{\text{cn}} \xrightarrow{\phi_{\parallel}} \mathbf{s}_q \llbracket \times \rrbracket^{\text{cn}}}, \quad \llbracket \bowtie \rrbracket^{\text{cn}} \simeq \boxed{\llbracket \times \rrbracket^{\text{cn}} \xrightarrow{\phi_{\times}} \mathbf{s}_q \llbracket \rangle \langle \rrbracket^{\text{cn}}}, \quad (2.17)$$

boxed diagrams being the filtration diagrams of the blob bimodules.

The disjoint union with vertical arcs on the left and on the right turns eq. (2.17) into filtration diagrams for the bimodules associated with diagrams (2.1):

$$\llbracket b_i^+ \rrbracket^{\text{cn}} \simeq \boxed{\llbracket \mathbb{1}_{(n)} \rrbracket^{\text{cn}} \xrightarrow{\phi_i} \mathbf{s}_q \llbracket s_i \rrbracket^{\text{cn}}}, \quad \llbracket b_i^- \rrbracket^{\text{cn}} \simeq \boxed{\llbracket s_i \rrbracket^{\text{cn}} \xrightarrow{\phi_i} \mathbf{s}_q \llbracket \mathbb{1}_{(n)} \rrbracket^{\text{cn}}}. \quad (2.18)$$

2.5.2. Filtered homomorphisms. The bimodules $\llbracket \bullet \bowtie \rrbracket$ and $\llbracket \bowtie \rrbracket$ are related to the diagonal bimodule $\llbracket \rangle \langle \rrbracket$ by two canonical morphisms χ_+ and χ_- associated with the filtration:

$$\begin{array}{ccc} \mathbf{s}_q \llbracket \rangle \langle \rrbracket & = & \mathbf{s}_q \llbracket \rangle \langle \rrbracket \\ \downarrow \chi_+ & & \downarrow \mathbb{1} \\ \llbracket \bullet \bowtie \rrbracket & \simeq & \boxed{\llbracket \times \rrbracket \xrightarrow{\phi_{\times}} \mathbf{s}_q \llbracket \rangle \langle \rrbracket} \end{array}, \quad \begin{array}{ccc} \llbracket \bullet \bowtie \rrbracket & \simeq & \boxed{\llbracket \rangle \langle \rrbracket \xrightarrow{\phi_{\parallel}} \mathbf{s}_q \llbracket \times \rrbracket} \\ \downarrow \chi_- & & \downarrow \mathbb{1} \\ \llbracket \rangle \langle \rrbracket & = & \llbracket \rangle \langle \rrbracket \end{array}$$

These morphisms appear in Rouquier complexes associated with braid group generators. Two other filtered homomorphisms relating blob bimodules appear later in our constructions:

$$\psi_- : \mathbf{s}_q \llbracket \bullet \bowtie \rrbracket \xrightarrow{y_- - x_-} \llbracket \bowtie \rrbracket, \quad \psi_+ : \mathbf{s}_q \llbracket \bowtie \rrbracket \xrightarrow{y_+ + x_+} \llbracket \bullet \bowtie \rrbracket.$$

Their action on filtered resolutions is depicted by the diagrams

$$\begin{array}{ccc} \mathbf{s}_q \llbracket \bullet \bowtie \rrbracket \simeq & \mathbf{s}_q \boxed{\llbracket \rangle \langle \rrbracket \xrightarrow{\phi_{\parallel}} \mathbf{s}_q \llbracket \times \rrbracket} & \mathbf{s}_q \llbracket \bowtie \rrbracket \simeq & \mathbf{s}_q \boxed{\llbracket \times \rrbracket \xrightarrow{\phi_{\times}} \mathbf{s}_q \llbracket \rangle \langle \rrbracket} \\ \downarrow \psi_- & \downarrow \mathbb{1} & \downarrow \psi_+ & \downarrow \mathbb{1} \\ \llbracket \bowtie \rrbracket \simeq & \boxed{\llbracket \times \rrbracket \xrightarrow{\phi_{\times}} \mathbf{s}_q \llbracket \rangle \langle \rrbracket} & \llbracket \bullet \bowtie \rrbracket \simeq & \boxed{\llbracket \rangle \langle \rrbracket \xrightarrow{\phi_{\parallel}} \llbracket \times \rrbracket} \end{array},$$

2.6. Filtered Rouquier complexes. Now we can add braid words to $\text{End}(n)$. We add $\llbracket \nearrow \rrbracket, \llbracket \searrow \rrbracket \in \text{End}(2)$ and these braids generate by composition and disjoint union the braid word monoid whose elements, by definition, are finite sequences of elementary positive and negative braids σ_i and σ_i^{-1} .

To elementary 2-strand braids we associate the complexes which are the cones of filtered-canonical morphisms within the category $\mathbf{Ch}(\mathbf{D}(\mathbb{Q}[\mathbf{x}, \mathbf{y}] - \mathbf{fi}, \mathbf{g}))$:

$$[\![\nearrow]\!] = \mathbf{t}_s \mathbf{a}_q \left[[\![\rangle]\!] \xrightarrow{\chi_+} \mathbf{t}_a^{-1} \mathbf{s}_q^{-1} [\![\bowtie]\!] \right] \cong \mathbf{t}_s \mathbf{a}_q \left[[\![\rangle]\!] \xrightarrow{\chi_+} \mathbf{t}_a^{-1} \boxed{\mathbf{s}_q^{-1} [\![\bowtie]\!] \xrightarrow{\phi_\times} [\![\rangle]\!] } \right], \quad (2.19)$$

$$[\![\searrow]\!] = (\mathbf{t}_s \mathbf{a}_q)^{-1} \left[\mathbf{t}_a [\![\bullet]\!] \xrightarrow{\chi_-} [\![\rangle]\!] \right] \cong (\mathbf{t}_s \mathbf{a}_q)^{-1} \left[\mathbf{t}_a \boxed{[\![\rangle]\!] \xrightarrow{\phi_\circ} \mathbf{s}_q [\![\bowtie]\!] } \xrightarrow{\chi_-} [\![\rangle]\!] \right]. \quad (2.20)$$

Recall that the boxes around the complexes indicate that the latter are ‘inner’ complexes, that is, they represent the objects of $\mathbf{D}(\mathbb{Q}[\mathbf{x}, \mathbf{y}] - \mathbf{fi}, \mathbf{g})$, while the complexes in square brackets are ‘outer’, that is, they are the complexes within the category of chain complexes $\mathbf{Ch}(\mathbf{D}(\mathbb{Q}[\mathbf{x}, \mathbf{y}] - \mathbf{fi}, \mathbf{g}))$.

2.7. HOMFLY-PT polynomial. In our conventions, in order to obtain the HOMFLY-PT polynomial of a link diagram as a graded Euler characteristic of its triply graded homology we introduce a factor

$$(-1)^{\deg_a} a^{\deg_a - \deg_t} q^{\deg_q}$$

multiplying the graded dimensions of the homology. As a result, the skein relation is

$$a^{1/2} q \langle \nearrow \searrow \rangle - a^{-1/2} q^{-1} \langle \searrow \nearrow \rangle = (q - q^{-1}) \langle \rangle \langle \rangle$$

and the HOMFLY-PT polynomial of the unknot is $(a^{1/2} q - a^{-1/2} q^{-1}) / (q - q^{-1})$.

2.8. Virtual relations. The virtual relations (2.2), (2.3) and (2.4) are satisfied after the application of the functor $[\![-]\!]$.

Theorem 2.2. *There are isomorphisms of modules*

$$[\![\bullet]\!] \cong [\![\bowtie]\!] \cong [\![\bowtie]\!], \quad [\![\bowtie]\!] \cong [\![\bowtie]\!] \cong [\![\bullet]\!], \quad (2.21)$$

and isomorphisms of complexes

$$[\![\bowtie]\!] \cong [\![\nearrow]\!], \quad [\![\bowtie]\!] \cong [\![\searrow]\!]. \quad (2.22)$$

Proof. Consider the presentation (2.17) of $[\![\bullet]\!]$ as a cone. In view of Remark 2.1 and a commutative diagram (2.15), a tensor product of this presentation with $[\![\bowtie]\!]$ over the intermediate variables turns it into the similar presentation for $[\![\bowtie]\!]$. This proves the left isomorphisms of (2.21). The right isomorphisms are proved similarly.

In order to prove the first isomorphism of (2.22), we sandwich the first presentation of $[\![\nearrow]\!]$ in (2.19) by two bimodules $[\![\bowtie]\!]$. Both bimodules of the cone remain the same and the

morphism χ_+ transforms into itself, because it is a canonical morphism associated with the filtration. The second isomorphism of (2.22) is proved similarly. \square

2.9. Morphism abbreviations within large diagrams. In order to avoid overloading of large diagrams appearing in sections 4, 5 and 6, we will omit labels on arrows connecting bimodules and their resolutions which correspond to two types of standard morphisms.

First, an unlabelled arrow connecting two bimodules (or their resolutions) of virtual braids, for example,

$$\llbracket \parallel \times \rrbracket \longrightarrow \llbracket \times \rrbracket$$

appearing in the diagram (5.16), corresponds to the virtual saddle (2.14).

Second, an unlabelled arrow connecting two bimodules (or their resolutions) of braid-graphs composed of black blobs, for example

$$\llbracket \times \bullet \times \rrbracket \longrightarrow \llbracket \parallel \bullet \rrbracket \quad (2.23)$$

appearing in the diagram (5.3) corresponds to the homomorphism χ_- which maps one of the blobs into two parallel arcs:

$$z \llbracket \times \bullet \parallel \rrbracket_y \otimes_{\mathbb{Q}[y]} y \llbracket \parallel \times \rrbracket_x \xrightarrow{\chi_- \otimes \mathbb{1}} z \llbracket \parallel \parallel \rrbracket_y \otimes_{\mathbb{Q}[y]} y \llbracket \parallel \times \rrbracket_x. \quad (2.24)$$

3. MULTI-FILTERED STRUCTURE OF SOERGEL BIMODULES AND ROUQUIER COMPLEXES

3.1. A category of multi-filtered semi-split chain complexes. Consider a lattice \mathbb{Z}^m . We denote its vertices as $v = (v_1, \dots, v_m)$. The vertices are partially ordered: $v' \geq v$ if $v'_i \geq v_i$ for all i . We define the total degree of a vertex as $|v| = v_1 + \dots + v_m$.

For an additive category \mathbf{A} we define an associated m -filtered semi-split category $\mathbf{A} - \mathbf{sg}^m$. Its objects are m -graded objects of \mathbf{A} , that is, an object of $\mathbf{A} - \mathbf{sg}^m$ is a *finite* direct sum of objects of \mathbf{A} : $A = \bigoplus_{v \in \mathbb{Z}^m} A_v$. The gradings define m filtrations $F^{(i)}$, $1 \leq i \leq m$ of the object A : $F_j^{(i)} A = \bigoplus_{v_i \geq j} A_v$. A morphism $A \xrightarrow{f} B$ between two objects of $\mathbf{A} - \mathbf{sg}^m$ is a morphism between them as objects of \mathbf{A} which respects all filtrations, that is, it has a nontrivial component $A_v \xrightarrow{f_{v,v'}} B_{v'}$ only when $v' \geq v$.

The category $\mathbf{A} - \mathbf{sg}^m$ is again additive, and we consider the category $\mathbf{Ch}^{[m]}(\mathbf{A})$ of bounded chain complexes over it. An object of this category is a bounded chain complex $(A, d) = (\dots \xrightarrow{d_{i-1}} A_i \xrightarrow{d_i} A_{i+1} \xrightarrow{d_{i+1}} \dots)$ whose chain objects A_i are finitely m -graded objects of \mathbf{A} and whose differentials are m -filtered morphisms. For $v \in \mathbb{Z}^m$ the complex A determines an associated graded complex

$$(A_v, d_v) = (\dots \rightarrow A_{i;v} \xrightarrow{d_{i;v}} A_{i+1;v} \rightarrow \dots),$$

which is an object in the category $\mathbf{Ch}(\mathbf{A})$ of chain complexes over \mathbf{A} . We refer to (A_v, d_v) as *constituent complexes* of (A, d) . We often drop the differential from the notation of the complex writing A instead of (A, d) .

3.2. An associated homotopy category. By the standard definition, if three morphisms $f, g, h \in \text{Hom}_{\mathbf{A}-\mathbf{sg}^m}(A, B)$ satisfy a relation $f - g = [d, h]$, then f and g are called homotopic. This defines the multi-filtered semi-split homotopy category $\mathbf{K}^{[m]}(\mathbf{A})$ over the additive category \mathbf{A} .

A filtration $F^{(i)}$ determines two endofunctors $F_j^{(i)}$ and $Q_j^{(i)}$ of the category $\mathbf{Ch}^{[m]}(\mathbf{A})$: $F_j^{(i)}A$ is a subcomplex of A , while $Q_j^{(i)}A$ is the corresponding quotient complex. These two complexes are related to A by *canonical* morphisms

$$F_j^{(i)}A \longrightarrow A \longrightarrow Q_j^{(i)}A. \quad (3.1)$$

Theorem 3.1. *The functors $F_j^{(i)}$ and $Q_j^{(i)}$ as well as the canonical morphisms (3.1) descend to the homotopy category $\mathbf{K}^{[m]}(\mathbf{A})$.*

The proof is standard (triangulated structure of homotopy category) and we leave it to the reader.

3.3. Multi-cone structure of multi-filtered semi-split chain complexes. Since the complex A is bounded, there exists a finite *domain* $\mathfrak{X}_A \subset \mathbb{Z}^m$ such that the complexes A_v are trivial unless $v \in \mathfrak{X}_A$. Note that a domain \mathfrak{X}_A is not determined by the complex A uniquely, because one can add to \mathfrak{X}_A more vertices v whose complexes A_v are trivial.

Define subsets of \mathbb{Z}^m :

$$\mathfrak{G}_i = \{v \in \mathbb{Z}^m : |v| = i\}, \quad \mathfrak{F}_i = \{v \in \mathbb{Z}^m : |v| \geq i\} = \bigcup_{j \geq i} \mathfrak{G}_j.$$

The total grading of an object of $\mathbf{A} - \mathbf{sg}^m$ is defined as the sum of individual m gradings: $G_i^{(\text{tot})}A = \bigoplus_{v \in \mathfrak{G}_i} A_v$. Then there is a forgetful functor $\mathcal{F}^{[\text{tot}]}: (\mathbf{A} - \mathbf{sg}^m) \rightarrow (\mathbf{A} - \mathbf{sg})$ from $\mathbf{A} - \mathbf{sg}^m$ to the filtered semi-split category $\mathbf{A} - \mathbf{sg} = \mathbf{A} - \mathbf{sg}^1$, which remembers only the total grading and filtration. This functor extends to the categories of complexes: $\mathcal{F}^{[\text{tot}]}: \mathbf{Ch}^{[m]}(\mathbf{A}) \rightarrow \mathbf{Ch}^{[1]}(\mathbf{A})$.

The total complex A is a multi-cone of associated graded complexes A_v : A is the direct sum of complexes $\bigoplus_{v \in \mathbb{Z}^m} A_v$ deformed by adding new differentials $d_{v,v'} = \sum_i d_{i,v,v'}$ for every pair $v' > v$. In particular, A is a multicone with respect to the total degree. Namely, define

$$F_i^{(\text{tot})}A = \left(\bigoplus_{v \in \mathfrak{F}_i} A_v, \sum_{v, v' \in \mathfrak{F}_i} d_{v,v'} \right)$$

Then there is a recursive cone relation

$$F_i^{(\text{tot})} A \cong \left[G_i^{(\text{tot})} A \xrightarrow{\sum_{v \in \mathfrak{G}_i, v' \in \mathfrak{F}_{i+1}} d_{v,v'}} F_{i+1}^{(\text{tot})} A \right]. \quad (3.2)$$

Theorem 3.2. *If the complexes A and B are homotopy equivalent in $\mathbf{Ch}^{[m]}(\mathbf{A})$, then*

- (1) *their associated graded complexes A_v and B_v are homotopy equivalent in $\mathbf{Ch}(\mathbf{A})$;*
- (2) *length-one differentials $A_v \xrightarrow{d_{A;v,v'}} A_{v'}$ and $B_v \xrightarrow{d_{B;v,v'}} B_{v'}$, where $|v' - v| = 1$, are homotopic as morphisms in $\mathbf{Ch}(\mathbf{A})$.*

The proof of this theorem is obvious and we leave it to the reader.

Theorem 3.3. *Let A be a complex in $\mathbf{Ch}^{[m]}(\mathbf{A})$ with a domain \mathfrak{X}_A . For a collection of complexes B_v , $v \in \mathfrak{X}_A$ such that $B_v \sim A_v$ in $\mathbf{Ch}(\mathbf{A})$ there exists a complex $B \sim A$ in $\mathbf{Ch}^{[m]}(\mathbf{A})$, whose constituent complexes are B_v .*

The proof of this theorem is based on a well-known lemma used to show that homotopy categories are triangulated.

Lemma 3.4. *For two pairs of homotopy equivalence complexes $A_i \sim B_i$, $i = 1, 2$ in $\mathbf{Ch}(\mathbf{A})$ and for a morphism $d_{12}^A: A_1 \rightarrow A_2$ there exists a morphism $d_{12}^B: B_1 \rightarrow B_2$ such that their cones are homotopy equivalent:*

$$\left[A_1 \xrightarrow{d_{12}^A} A_2 \right] \sim \left[B_1 \xrightarrow{d_{12}^B} B_2 \right].$$

Proof. Denote $A = A_1 \oplus A_2$, $d_A = d_{A_1} + d_{A_2}$ and similarly for B . By the assumption of the lemma there is a homotopy equivalence

$$h_A \circlearrowleft (A, d_A) \begin{array}{c} \xrightarrow{f_{AB}} \\ \xleftarrow{f_{BA}} \end{array} (B, d_B) \circlearrowright h_B$$

Choose $d_{12}^B = f_{AB} d_{12}^A f_{BA}$, then a straightforward calculation shows that the homotopy equivalence

$$H_A \circlearrowleft (A, d_A + d_{12}^A) \begin{array}{c} \xrightarrow{F_{AB}} \\ \xleftarrow{F_{BA}} \end{array} (B, d_B + d_{12}^B) \circlearrowright H_B$$

is established by the morphisms

$$\begin{aligned}
F_{AB} &= f_{AB} + f_{AB} d_{12}^A h_A + (h_B f_{AB} - f_{AB} h_A) d_{12}^A, \\
F_{BA} &= f_{BA} + h_A d_{12}^A f_{BA} + d_{12}^A (f_{BA} h_B - h_A f_{BA}), \\
H_A &= h_A + f_{BA} (h_B f_{AB} - f_{AB} h_A) + h_A d_{12}^A h_A - d_{12}^A (h_A)^2, \\
H_B &= h_B.
\end{aligned} \tag{3.3}$$

□

Proof of Theorem 3.3. Let $i_{\min} = \min\{|v|, v \in \mathfrak{X}_A\}$ and $i_{\max} = \max\{|v|, v \in \mathfrak{X}_A\}$. We will prove the theorem for all subcomplexes $F_i^{(\text{tot})} A$ with domains $\mathfrak{X}_{A;i} = \mathfrak{X}_A \cap \mathfrak{F}_i$ by induction from $i = i_{\max} + 1$ to $i = i_{\min}$ by going over i ‘backwards’.

If $i = i_{\max} + 1$, then $\mathfrak{X}_{A;i} = \emptyset$, the complex $F_i^{(\text{tot})} A$ is trivial and the claim of the theorem is obvious. If $i = i_{\min}$, then $\mathfrak{X}_{A;i} = \mathfrak{X}_A$, $F_i^{(\text{tot})} A = A$ and the veracity of the theorem for $F_i^{(\text{tot})} A$ implies that it holds for A . Now assume that for some value of i the theorem holds for $F_{i+1}^{(\text{tot})} A$. Its extension to $F_i^{(\text{tot})} A$ is performed with the help of eq.(3.2) and Lemma 3.4 by observing that the differentials, homotopy equivalences and homotopies defined explicitly by the formulas (3.3) are compositions of multi-filtered morphisms, so they are multi-filtered themselves and define homotopy equivalence in the category $\mathbf{Ch}^{[m]}(\mathbf{A})$. □

From now on till the end of this subsection we make a distinction between the complex (A, d) , which is an object of $\mathbf{Ch}^{[m]}(\mathbf{A})$ and the sum of its chain objects A , which is an object of $\mathbf{A} - \mathbf{sg}^m$ endowed with an extra (homological) grading. Any endomorphism h of A can be split according to the total degree:

$$h = \sum_{i=0}^{\infty} h_{(i)}, \quad h_{(i)} = \sum_{\substack{v, v' \in \mathfrak{X}_A \\ |v' - v| = i}} h_{v, v'}.$$

In particular, we split the differential d of the complex (A, d) : $d = \sum_{i=0}^{\infty} d_{(i)}$. Obviously, $d_{(0)} = \sum_{v \in \mathfrak{X}_A} d_v$ and $(A, d_{(0)}) = \bigoplus_{v \in \mathfrak{X}_A} (A_v, d_v)$. The condition $d^2 = 0$ implies relations

$$d_{(i)}^2 = - \sum_{j=0}^{i-1} [d_{(j)}, d_{(2i-j)}], \quad \sum_{j=0}^i [d_{(j)}, d_{(2i+1-j)}] = 0. \tag{3.4}$$

These relations at $i = 1$ say $[d_{(0)}, d_{(1)}] = 0$, $d_{(1)}^2 = -[d_{(0)}, d_{(2)}]$, so $d_{(1)}$ is an element of $\text{End}_{\mathbf{Ch}^{[m]}(\mathbf{A})}(A, d_{(0)})$ with homological degree one and with the property that it is closed and its square is homotopic to zero.

3.4. Soergel bimodules and multi-filtered semi-split chain complexes. For the category $\mathbb{Q}[\mathbf{x}, \mathbf{y}] - \mathbf{fr}, \mathbf{g}$ of free q -graded $\mathbb{Q}[\mathbf{x}, \mathbf{y}]$ -modules, consider an associated category of

multi-filtered semi-split chain complexes $\mathbf{Ch}^{[m]}(\mathbb{Q}[\mathbf{x}, \mathbf{y}] - \mathbf{fr}, \mathbf{g})$. There is a functor $\mathcal{F}_D^{[\text{tot}]}$

$$\begin{array}{ccc}
 & \mathcal{F}_D^{[\text{tot}]} & \\
 & \curvearrowright & \\
 \mathbf{Ch}^{[m]}(\mathbb{Q}[\mathbf{x}, \mathbf{y}] - \mathbf{fr}, \mathbf{g}) & \xrightarrow{\mathcal{F}^{[\text{tot}]}} & \mathbf{Ch}^{[1]}(\mathbb{Q}[\mathbf{x}, \mathbf{y}] - \mathbf{fr}, \mathbf{g}) \\
 \downarrow & & \downarrow \\
 \mathbf{K}^{[m]}(\mathbb{Q}[\mathbf{x}, \mathbf{y}] - \mathbf{fr}, \mathbf{g}) & \xrightarrow{\mathcal{F}^{[\text{tot}]}} & \mathbf{K}^{[1]}(\mathbb{Q}[\mathbf{x}, \mathbf{y}] - \mathbf{fr}, \mathbf{g}) \longrightarrow \mathbf{D}(\mathbb{Q}[\mathbf{x}, \mathbf{y}] - \mathbf{fr}, \mathbf{g})
 \end{array} \quad (3.5)$$

which factors through the homotopy categories. The functor \mathcal{R} is surjective and if $\mathcal{R}(A) = B$, then A is a free resolution of the object B in the derived category of filtered modules.

Recall that a $\mathbb{Q}[\mathbf{x}, \mathbf{y}]$ -module $[[s]]$ of eq. (2.6) corresponding to a permutation $s \in S_n$, has a canonical Koszul resolution $[[s]]^{\text{cn}}$ of eq. (2.9), which is an object in $\mathbf{Ch}(\mathbb{Q}[\mathbf{x}, \mathbf{y}] - \mathbf{fr}, \mathbf{g})$. To a braid-graph $\gamma = \gamma_m \cdots \gamma_1$ presented as a product of m elementary braid-graphs of the type (2.1) we will associate a canonical complex $[[\gamma]]^{\text{cn}}$ from $\mathbf{Ch}^{[m]}(\mathbb{Q}[\mathbf{x}, \mathbf{y}] - \mathbf{fr}, \mathbf{g})$ such that $\mathcal{F}_D^{[\text{tot}]}[[\gamma]]^{\text{cn}} \cong [[\gamma]]$, that is, $\mathcal{F}^{[\text{tot}]}[[\gamma]]^{\text{cn}}$ is a free filtered resolution of $[[\gamma]]$.

As a first step towards $[[\gamma]]^{\text{cn}}$ we define a different complex $[[\gamma]]^\otimes$ from $\mathbf{Ch}^{[m]}(\mathbb{Q}[\mathbf{x}, \mathbf{y}] - \mathbf{fr}, \mathbf{g})$ with the same property $\mathcal{F}^{[\text{tot}]}[[\gamma]]^\otimes \cong [[\gamma]]$. If γ is an elementary braid-graph (2.1), then $m = 1$ and we define $[[\gamma]]^\otimes = [[\gamma]]^{\text{cn}}$, where $[[\gamma]]^{\text{cn}}$ is defined by eq. (2.18) and the single filtration of $\mathbf{Ch}^{[1]}$ is the filtration of $[[\gamma]]^{\text{cn}}$. For a general $\gamma = \gamma_m \cdots \gamma_1$ we define $[[\gamma]]^\otimes$ as the product of complexes $[[\gamma_i]]^{\text{cn}}$ over the intermediate variables, the i -th filtration of $\mathbf{Ch}^{[m]}$ being the filtration of the i -th factor $[[\gamma_i]]^{\text{cn}}$.

Here is a simple example of $[[\gamma]]^\otimes$ for $m = 2$:

$$\begin{array}{ccc}
 \mathbf{z} \left[\begin{array}{c} \bullet \\ \circ \end{array} \right]_{\mathbf{x}}^\otimes & = & \boxed{ \begin{array}{ccc} \mathbf{z} \left[\right]_{\mathbf{y}}^{\text{cn}} \otimes_{\mathbb{Q}[\mathbf{y}]} \mathbf{y} \left[\times \right]_{\mathbf{x}}^{\text{cn}} & \xrightarrow{\phi_1 \otimes 1} & \mathbf{s}_q \mathbf{z} \left[\times \right]_{\mathbf{y}}^{\text{cn}} \otimes_{\mathbb{Q}[\mathbf{y}]} \mathbf{y} \left[\times \right]_{\mathbf{x}}^{\text{cn}} \\ \downarrow 1 \otimes \phi_x & & \downarrow 1 \otimes \phi_x \\ \mathbf{s}_q \mathbf{z} \left[\right]_{\mathbf{y}}^{\text{cn}} \otimes_{\mathbb{Q}[\mathbf{y}]} \mathbf{y} \left[\right]_{\mathbf{x}}^{\text{cn}} & \xrightarrow{-\phi_1 \otimes 1} & \mathbf{s}_q^2 \mathbf{z} \left[\times \right]_{\mathbf{y}}^{\text{cn}} \otimes_{\mathbb{Q}[\mathbf{y}]} \mathbf{y} \left[\right]_{\mathbf{x}}^{\text{cn}} \end{array} } \quad (3.6)
 \end{array}$$

The constituent complexes $[[\gamma]]_v^\otimes$ of $[[\gamma]]^\otimes$ sit at the vertices of the unit m -dimensional cube $\mathfrak{C}^m \subset \mathbb{Z}^m$ and they are tensor products of canonical resolutions appearing in the cones (2.18) shifted by $\mathbf{s}_q^{|v|}$. The coordinates of a vertex v of the cube determine which of the complexes $[[1_{(n)}]]^{\text{cn}}$ or $[[s_i]]^{\text{cn}}$ appear at each position in this tensor product. Let $d^\otimes = d_{(0)}^\otimes + d_{(1)}^\otimes$ be the differential of the complex $[[\gamma]]^\otimes$ expanded according to the total degree. $d_{(0)}^\otimes$ is the sum of differentials of the constituent complexes $[[\gamma]]_v^\otimes$, while $d_{(1)}^\otimes = \sum_e d_e^\otimes$ is the sum of components

corresponding to the edges of the cube \mathfrak{C}^m having the form $d_e^\otimes = \pm \mathbb{1}^{\otimes i} \otimes \phi \otimes \mathbb{1}^{\otimes (m-i-1)}$, where ϕ is a virtual saddle morphism ϕ_{\parallel} or ϕ_{\times} .

Let s_v be the product of permutations $\mathbb{1}_{(n)}$ or s_i as they appear in the tensor product of complexes $[\mathbb{1}_{(n)}]^{\text{cn}}$ and $[s_i]^{\text{cn}}$ at v . Then there is a homotopy equivalence $[\gamma]_v^\otimes \sim [s_v]^{\text{cn}}$ since, in view of eq.(2.7), both complexes are resolutions of the bimodule $[s_v]$. Hence, according to Theorem 3.3, the complex $[\gamma]^\otimes$ is homotopy equivalent within the category $\mathbf{Ch}^{[m]}(\mathbb{Q}[\mathbf{x}, \mathbf{y}] - \mathbf{fr}, \mathbf{g})$ to a complex $[\gamma]^{\text{cn}}$, whose constituent complexes are $\mathbf{s}_q^{[v]}[s_v]^{\text{cn}}$ and whose differentials of positive total degree are determined by the choice of contraction homotopy. In particular, the complex (3.6) transforms into the complex (3.7).

Let $d = \sum_{i=0}^m d_{(i)}$ be the differential of the complex $[\gamma]^{\text{cn}}$ expanded according to the total degree. The component $d_{(0)}$ is the sum of the differentials of the canonical complexes $[s_v]^{\text{cn}}$, hence it is predetermined. The component $d_{(1)}$ is the sum of components corresponding to the edges of the cube \mathfrak{C}^m : $d_{(1)} = \sum_e d_e$. If two vertices $v, v' \in \mathfrak{C}^m$ are connected by an edge e , then there exists a transposition $s_{(e)}$ such that $s_{v'} = s_v s_{(e)}$ and in view of the commutative diagram (2.15) and Theorem 3.2, after the contraction of constituent complexes the differential of the edge e becomes (up to a sign) homotopic to the virtual saddle morphism $\phi_{(e)}$ between the resolutions: $d_e \sim \pm \phi_{(e)}$. The contraction might also generate longer differentials $d_{(i)}$, $i \geq 2$ in $[\gamma]^{\text{cn}}$. We show that the total degree one differentials of $[\gamma]^{\text{cn}}$ are (up to a sign) precisely the virtual saddle morphisms $\phi_{(e)}$: $d_e = \pm \phi_{(e)}$ and differentials of higher total degree are absent.

Theorem 3.5. *The differential d of the complex $[\gamma]^{\text{cn}}$ has the following canonical form: $d_e = \pm \phi_{(e)}$ and $d_{(i)} = 0$ for $i \geq 2$.*

For example, according to this theorem,

$$\left[\begin{array}{c} \text{diagram of a crossing with a dot} \end{array} \right]^{\text{cn}} = \begin{array}{ccc} \left[\begin{array}{c} \text{diagram of a crossing} \end{array} \right]^{\text{cn}} & \xrightarrow{\phi_{\times}} & \mathbf{s}_q \left[\begin{array}{c} \text{diagram of a crossing} \end{array} \right]^{\text{cn}} \\ \phi_{\parallel} \downarrow & & \downarrow \phi_{\parallel} \\ \mathbf{s}_q \left[\begin{array}{c} \text{diagram of a crossing} \end{array} \right]^{\text{cn}} & \xrightarrow{-\phi_{\times}} & \mathbf{s}_q^2 \left[\begin{array}{c} \text{diagram of a crossing} \end{array} \right]^{\text{cn}} \end{array} \quad (3.7)$$

The proof of Theorem 3.5 is based on the following lemma.

Lemma 3.6. *If h is a $\mathbb{Q}[\mathbf{x}, \mathbf{y}]$ -endomorphism of the complex $[\gamma]^{\text{cn}}$ and h is homogeneous with respect to a -degree, q -degree and total degree, then*

$$\deg_q h \geq 2(\deg_{[\text{tot}]} h + \deg_a h). \quad (3.8)$$

Proof. It is easy to see that the Koszul resolution $[[s]]^{\text{cn}}$ of eq.(2.9) has the form

$${}_y[[s]]_x^{\text{cn}} = (\mathbf{a}_q^m F_m \rightarrow \cdots \rightarrow \mathbf{a}_q^i F_i \rightarrow \cdots \rightarrow F_0), \quad \text{where} \quad F_i = \underbrace{\mathbb{Q}[\mathbf{x}, \mathbf{y}] \oplus \cdots \oplus \mathbb{Q}[\mathbf{x}, \mathbf{y}]}_{\binom{m}{i} \text{ times}}.$$

Therefore, for any homogeneous homomorphism $h \in \text{Hom}_{\mathbb{Q}[\mathbf{x}, \mathbf{y}]}({}_y[[s]]_x^{\text{cn}}, {}_y[[s']]_x^{\text{cn}})$, there is a bound $\deg_q h \geq 2 \deg_a h$. Hence for any homogeneous homomorphism

$$h \in \text{Hom}_{\mathbb{Q}[\mathbf{x}, \mathbf{y}]}(\mathbf{s}_q^{|v|}[[s_v]]^{\text{cn}}, \mathbf{s}_q^{|v'|}[[s_{v'}]]^{\text{cn}}) \quad (3.9)$$

between two constituent complexes of $[[\gamma]]^{\text{cn}}$ there is a bound

$$\deg_q h \geq 2(|v' - v| + \deg_a h), \quad (3.10)$$

and it implies the bound (3.8). \square

Proof of Theorem 3.5. Since $d_e \sim \pm \phi_{(e)}$, there should exist a homomorphism (3.9), v and v' being the endpoints of e , such that

$$d_e = \pm \phi_{(e)} + d_{v'} h + h d_v, \quad \deg_q h = 0, \quad \deg_a h = 0,$$

Since $|v' - v| = 1$, the degree requirements on h are incompatible with the bound (3.10), hence $h = 0$ and $d_e = \pm \phi_{(e)}$.

A component $d_{(i)}$ of the differential d of the complex $[[\gamma]]^{\text{cn}}$ is a homogeneous endomorphism of $[[\gamma]]^{\text{cn}}$ with degrees

$$\deg_q d_{(i)} = 0, \quad \deg_a d_{(i)} = -1, \quad \deg_{[\text{tot}]} d_{(i)} = i,$$

and if $i \geq 2$, then they are incompatible with the bound (3.8), which means that $d_{(i)} = 0$. \square

3.5. Multi-filtered Rouquier complexes. For an additive category \mathbf{A} and for a positive integer m we define a category of multi-complexes $\mathbf{Ch}^m(\mathbf{A})$. Its objects are pairs (A, d) , where A is an object of the m -graded category over \mathbf{A} : $A = \bigoplus_{v \in \mathfrak{X}_A} A_v$, where $\mathfrak{X}_A \subset \mathbb{Z}^m$ is a finite subset, $\deg_{\text{tot}} = \deg_1 + \cdots + \deg_m$ being the homological degree, while $d \in \text{End}_{\mathbf{A}}(A)$, $d = d_1 + \cdots + d_m$, $\deg_i d_j = \delta_{ij}$, where δ_{ij} is the Kronecker symbol and $[d_i, d_j] = 0$. For our present purposes we don't have to define morphisms in $\mathbf{Ch}^m(\mathbf{A})$. There is an obvious forgetful functor $\mathcal{F}^{\text{tot}}: \mathbf{Ch}^m(\mathbf{A}) \rightarrow \mathbf{Ch}(\mathbf{A})$. A combination of this functor and the forgetful functor $\mathcal{F}_{\mathbf{D}}^{[\text{tot}]}$ of eq.(3.5) yields a forgetful functor

$$\begin{array}{ccc} \mathbf{Ch}^m(\mathbf{Ch}^{[m]}(\mathbb{Q}[\mathbf{x}, \mathbf{y}] - \mathbf{fr}, \mathbf{g})) & & \\ \downarrow \mathcal{F}^{\text{tot}} & \searrow \mathcal{F}_{\mathbf{D}}^{\text{tot}} & \\ \mathbf{Ch}(\mathbf{Ch}^{[m]}(\mathbb{Q}[\mathbf{x}, \mathbf{y}] - \mathbf{fr}, \mathbf{g})) & \xrightarrow{\mathcal{F}_{\mathbf{D}}^{[\text{tot}]}} & \mathbf{Ch}(\mathbf{D}(\mathbb{Q}[\mathbf{x}, \mathbf{y}] - \mathbf{fi}, \mathbf{g})) \end{array}$$

which factors through the category $\mathbf{Ch}(\mathbf{Ch}^{[m]}(\mathbb{Q}[\mathbf{x}, \mathbf{y}] - \mathbf{fr}, \mathbf{g}))$.

For a braid word β we define a canonical object $[\![\beta]\!]^{\text{cn}}$ of the category $\mathbf{Ch}^m(\mathbf{Ch}^{[m]}(\mathbb{Q}[\mathbf{x}, \mathbf{y}] - \mathbf{fr}, \mathbf{g}))$ such that $\mathcal{F}_{\mathbf{D}}^{\text{tot}}[\![\beta]\!]^{\text{cn}} \cong [\![\beta]\!]$. First, we define the object $[\![\beta]\!]^{\otimes}$ in the same category by taking the tensor product of Rouquier complexes (2.19) and (2.20) over the algebras of intermediate variables selected in accordance with the content of β . The degree of each complex is considered as independent, so the result (up to an overall shift in multi-degree) is the complex supported in the unit cube $\mathfrak{C}^m \subset \mathbb{Z}^m$.

The coordinates 0 and 1 in an i -th coordinate direction correspond to a replacement of the i -th elementary factor in the braid word β by either the identity braid $\mathbb{1}_{(n)}$ or by a blob generator b_j^+ or b_j^- , where j is the index of the left strand participating in the i -th elementary crossing. To each vertex $v \in \mathfrak{C}^m$ corresponds a braid-graph γ_v , which is the composition of identity braids and blob generators selected according to its coordinates. A constituent object at v is the complex $[\![\gamma_v]\!]^{\otimes}$. At an edge e connecting the vertices γ_v and $\gamma_{v'}$ sits a component of the differential $d_e^{\text{cn}}: [\![\gamma_v]\!]^{\otimes} \rightarrow [\![\gamma_{v'}]\!]^{\otimes}$, which is a canonical morphism corresponding to the i -th filtration.

In order to obtain the canonical complexes $[\![\beta]\!]^{\text{cn}}$ we replace the constituent objects $[\![\gamma_v]\!]^{\otimes}$ with homotopy equivalent objects $[\![\gamma_v]\!]^{\text{cn}}$. According to Theorem 3.1, the definition of a canonical morphism extends to the homotopy category, so the differential d_e^{cn} at the edge e is still the canonical morphism associated with i -th filtration.

The following example illustrates the structure of the canonical complex $[\![\beta]\!]^{\text{cn}}$:

$$[\![\text{crossing}]\!]^{\text{cn}} = \left[\begin{array}{c} \left[\begin{array}{c} \left[\text{blob}^+ \right]^{\text{cn}} \longrightarrow \left[\text{crossing} \right]^{\text{cn}} \\ \left[\text{blob}^- \right]^{\text{cn}} \longrightarrow \left[\text{crossing} \right]^{\text{cn}} \end{array} \right] \end{array} \right] = \left[\begin{array}{c} \left[\begin{array}{c} \left[\text{cap} \right]^{\text{cn}} \longrightarrow \left[\text{crossing} \right]^{\text{cn}} \\ \left[\text{cup} \right]^{\text{cn}} \longrightarrow \left[\text{crossing} \right]^{\text{cn}} \end{array} \right] \longrightarrow \left[\text{crossing} \right]^{\text{cn}} \end{array} \right] \quad (3.11)$$

We omitted all shift functors in order to emphasize the structure of constituent complexes $[\![\gamma_v]\!]^{\text{cn}}$.

4. SECOND REIDEMEISTER MOVE

Theorem 4.1. *The following complexes are homotopy equivalent in $\mathbf{Ch}(\mathbf{D}(\mathbb{Q}[\mathbf{x}, \mathbf{y}] - \mathbf{fi}, \mathbf{g}))$:*

$$\left[\text{crossing} \right] \sim \left[\text{cap} \right], \quad \left[\text{crossing} \right] \sim \left[\text{cup} \right]. \quad (4.1)$$

The proof is based on the lemma:

Lemma 4.2. *The ‘double blob’ bimodules split;*

$$\left[\begin{array}{c} \bullet \\ \diagup \diagdown \\ \bullet \end{array} \right] \cong \left[\begin{array}{c} \diagup \diagdown \\ \bullet \end{array} \right] \oplus s_q \left[\begin{array}{c} \bullet \\ \diagup \diagdown \\ \bullet \end{array} \right], \quad \left[\begin{array}{c} \bullet \\ \diagdown \diagup \\ \bullet \end{array} \right] \cong \left[\begin{array}{c} \diagdown \diagup \\ \bullet \end{array} \right] \oplus s_q \left[\begin{array}{c} \bullet \\ \diagdown \diagup \\ \bullet \end{array} \right],$$

and the following diagrams are commutative:

$$\begin{array}{ccc} \left[\begin{array}{c} \bullet \\ \diagup \diagdown \\ \bullet \end{array} \right] & \xrightarrow{\chi_+ \otimes \mathbb{1}} & \left[\begin{array}{c} \bullet \\ \diagup \diagdown \\ \bullet \end{array} \right] & \xrightarrow{\mathbb{1} \otimes \chi_-} & \left[\begin{array}{c} \diagup \diagdown \\ \bullet \end{array} \right] & \left[\begin{array}{c} \bullet \\ \diagdown \diagup \\ \bullet \end{array} \right] & \xrightarrow{\mathbb{1} \otimes \chi_+} & \left[\begin{array}{c} \bullet \\ \diagdown \diagup \\ \bullet \end{array} \right] & \xrightarrow{\chi_- \otimes \mathbb{1}} & \left[\begin{array}{c} \diagdown \diagup \\ \bullet \end{array} \right] \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \left[\begin{array}{c} \bullet \\ \diagup \diagdown \\ \bullet \end{array} \right] & \xrightarrow{\begin{pmatrix} 0 \\ \mathbb{1} \end{pmatrix}} & \left[\begin{array}{c} \diagup \diagdown \\ \bullet \end{array} \right] \oplus s_q \left[\begin{array}{c} \bullet \\ \diagup \diagdown \\ \bullet \end{array} \right] & \xrightarrow{(\mathbb{1} \ \psi_-)} & \left[\begin{array}{c} \diagup \diagdown \\ \bullet \end{array} \right] & \left[\begin{array}{c} \bullet \\ \diagdown \diagup \\ \bullet \end{array} \right] & \xrightarrow{\begin{pmatrix} 0 \\ \mathbb{1} \end{pmatrix}} & \left[\begin{array}{c} \diagdown \diagup \\ \bullet \end{array} \right] \oplus s_q \left[\begin{array}{c} \bullet \\ \diagdown \diagup \\ \bullet \end{array} \right] & \xrightarrow{(\mathbb{1} \ \psi_-)} & \left[\begin{array}{c} \diagdown \diagup \\ \bullet \end{array} \right] \end{array} \quad (4.2)$$

Proof. We will prove only the first splitting, the second one is proved similarly. We represent the *l.h.s.* bimodule by its canonical resolution (3.7) and then replace it by the isomorphic complex which splits:

$$\begin{array}{ccc} \left[\begin{array}{c} \bullet \\ \diagup \diagdown \\ \bullet \end{array} \right]^{\text{cn}} & \cong & \boxed{\left[\begin{array}{c} \diagup \diagdown \\ \bullet \end{array} \right]^{\text{cn}} \xrightarrow{\begin{pmatrix} \phi \\ \phi \end{pmatrix}} s_q(\left[\begin{array}{c} \diagup \diagdown \\ \bullet \end{array} \right]^{\text{cn}} \oplus \left[\begin{array}{c} \diagdown \diagup \\ \bullet \end{array} \right]^{\text{cn}}) \xrightarrow{(\phi - \phi)} s_q^2 \left[\begin{array}{c} \diagup \diagdown \\ \bullet \end{array} \right]^{\text{cn}}} \\ \downarrow \cong & & \downarrow \mathbb{1} \quad \downarrow \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad \downarrow \mathbb{1} \\ \left[\begin{array}{c} \diagup \diagdown \\ \bullet \end{array} \right]^{\text{cn}} \oplus s_q \left[\begin{array}{c} \bullet \\ \diagup \diagdown \\ \bullet \end{array} \right]^{\text{cn}} & \cong & \boxed{\left[\begin{array}{c} \diagup \diagdown \\ \bullet \end{array} \right]^{\text{cn}} \xrightarrow{\begin{pmatrix} 0 \\ \phi \end{pmatrix}} s_q(\left[\begin{array}{c} \diagup \diagdown \\ \bullet \end{array} \right]^{\text{cn}} \oplus \left[\begin{array}{c} \diagdown \diagup \\ \bullet \end{array} \right]^{\text{cn}}) \xrightarrow{(\phi \ 0)} s_q^2 \left[\begin{array}{c} \diagup \diagdown \\ \bullet \end{array} \right]^{\text{cn}}} \end{array} \quad (4.3)$$

The lower lines of the diagrams (4.2) are established by a straightforward computation. \square

Proof of Theorem 4.1. We prove only the first homotopy equivalence of (4.1), the second one is proved similarly. The proof is well-known, we just have to verify that the isomorphisms and homotopies appearing there are filtered:

$$\left[\begin{array}{c} \diagup \diagdown \\ \bullet \end{array} \right] \cong \left[\begin{array}{c} \left[\begin{array}{c} \bullet \\ \diagup \diagdown \\ \bullet \end{array} \right] \xrightarrow{\mathbb{1} \otimes \chi_-} \left[\begin{array}{c} \diagup \diagdown \\ \bullet \end{array} \right] \\ \chi_+ \otimes \mathbb{1} \downarrow \quad \downarrow \chi_+ \otimes \mathbb{1} \\ s^{-1} \left[\begin{array}{c} \bullet \\ \diagup \diagdown \\ \bullet \end{array} \right] \xrightarrow{-\mathbb{1} \otimes \chi_-} s^{-1} \left[\begin{array}{c} \diagup \diagdown \\ \bullet \end{array} \right] \end{array} \right] \cong \left[\begin{array}{c} \left[\begin{array}{c} \bullet \\ \diagup \diagdown \\ \bullet \end{array} \right] \xrightarrow{\chi_-} \left[\begin{array}{c} \diagup \diagdown \\ \bullet \end{array} \right] \\ \begin{pmatrix} 0 \\ \mathbb{1} \end{pmatrix} \downarrow \quad \downarrow \chi_+ \\ s^{-1} \left[\begin{array}{c} \diagup \diagdown \\ \bullet \end{array} \right] \oplus \left[\begin{array}{c} \bullet \\ \diagup \diagdown \\ \bullet \end{array} \right] \xrightarrow{(\mathbb{1} \ \psi_-)} s^{-1} \left[\begin{array}{c} \diagup \diagdown \\ \bullet \end{array} \right] \end{array} \right] \sim \left[\begin{array}{c} \diagup \diagdown \\ \bullet \end{array} \right].$$

The last homotopy comes from contracting the subcomplex $s^{-1} \left[\begin{array}{c} \diagup \diagdown \\ \bullet \end{array} \right] \xrightarrow{\mathbb{1}} s^{-1} \left[\begin{array}{c} \diagup \diagdown \\ \bullet \end{array} \right]$ and then contracting the quotient complex $\left[\begin{array}{c} \bullet \\ \diagup \diagdown \\ \bullet \end{array} \right] \xrightarrow{\mathbb{1}} \left[\begin{array}{c} \diagup \diagdown \\ \bullet \end{array} \right]$. Both complexes are filtered contractible.

\square

5. THIRD REIDEMEISTER MOVE

Theorem 5.1. *There exists a homotopy equivalence*

$$\left[\left[\begin{array}{c} \text{diagram} \end{array} \right] \right] \xrightleftharpoons[f_{r1}]{f_{lr}} \left[\left[\begin{array}{c} \text{diagram} \end{array} \right] \right] \quad (5.1)$$

such that the homotopy equivalence morphisms f_{lr} and f_{r1} violate filtration by at most two units.

Proof. The general strategy is standard. The braids of eq.(5.1) are related by the reflection symmetry with respect to the vertical axis. We construct a special reflection symmetric complex C_{sym} and establish two homotopy equivalences

$$\left[\left[\begin{array}{c} \text{diagram} \end{array} \right] \right] \xrightleftharpoons[f_{sl}]{f_{ls}} C_{\text{sym}} \xrightleftharpoons[f_{rs}]{f_{sr}} \left[\left[\begin{array}{c} \text{diagram} \end{array} \right] \right] , \quad (5.2)$$

such that f_{sl} and f_{sr} are filtered, while f_{ls} and f_{rs} violate filtration by two units. By symmetry, it is sufficient to establish only the left homotopy equivalence, and the claim of the theorem follows.

Consider the left complex in the diagram (5.2):

$$\begin{aligned} \left[\left[\begin{array}{c} \text{diagram} \end{array} \right] \right] &= \left[\begin{array}{c} \text{complex diagram} \end{array} \right] \\ &= \left[\left[\begin{array}{c} \text{diagram} \end{array} \right] \xrightarrow{\mathbb{1} \otimes \chi_- \otimes \mathbb{1}} \left[\begin{array}{c} \text{diagram} \end{array} \right] \right] \end{aligned} \quad (5.3)$$

Here unlabelled arrows correspond to the morphisms χ_- of the type (2.24). The cone in the last line comes from splitting the cube complex into the back face and the front face. The second complex of the cone can be simplified with the help of two lemmas. The first one is the analog of Lemma 4.2:

Lemma 5.2. *The following filtered bimodule splits:*

$$\left[\left[\begin{array}{c} \text{diagram} \end{array} \right] \right] \cong \left[\left[\begin{array}{c} \text{diagram} \end{array} \right] \right] \oplus \mathbf{s}_q \left[\left[\begin{array}{c} \text{diagram} \end{array} \right] \right], \quad (5.4)$$

and the following diagram is commutative:

$$\begin{array}{ccccc}
 \left[\left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right] \right] & \xleftarrow{\chi_- \otimes \mathbb{1}} & \left[\left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right] \right] & \xrightarrow{\mathbb{1} \otimes \chi_-} & \left[\left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right] \right] \\
 \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
 \left[\left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right] \right] & \xleftarrow{(\mathbb{1} \ \psi_+)} & \left[\left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right] \right] \oplus \mathbf{s}_q \left[\left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right] \right] & \xrightarrow{(\mathbb{1} \ 0)} & \left[\left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right] \right]
 \end{array}$$

Proof. The proof is similar to that of Lemma 4.2, the analog of the diagram (4.3) being

$$\begin{array}{ccc}
 \left[\left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right] \right]^{\text{cn}} & \cong & \boxed{\left[\right] \left[\right]^{\text{cn}} \xrightarrow{\begin{pmatrix} \phi \\ \phi \end{pmatrix}} \mathbf{s}_q \left(\left[\left[\begin{array}{c} \diagup \quad \diagdown \end{array} \right] \right]^{\text{cn}} \oplus \left[\left[\begin{array}{c} \diagup \quad \diagdown \end{array} \right] \right]^{\text{cn}} \right) \xrightarrow{\begin{pmatrix} \phi & -\phi \end{pmatrix}} \mathbf{s}_q^2 \left[\right] \left[\right]^{\text{cn}}} \\
 \downarrow \cong & & \downarrow \mathbb{1} \qquad \downarrow \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \qquad \downarrow \mathbb{1} \\
 \left[\left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right] \right]^{\text{cn}} \oplus \mathbf{s}_q \left[\left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right] \right]^{\text{cn}} & \cong & \boxed{\left[\right] \left[\right]^{\text{cn}} \xrightarrow{\begin{pmatrix} 0 \\ \phi \end{pmatrix}} \mathbf{s}_q \left(\left[\left[\begin{array}{c} \diagup \quad \diagdown \end{array} \right] \right]^{\text{cn}} \oplus \left[\left[\begin{array}{c} \diagup \quad \diagdown \end{array} \right] \right]^{\text{cn}} \right) \xrightarrow{\begin{pmatrix} \phi & 0 \end{pmatrix}} \mathbf{s}_q^2 \left[\right] \left[\right]^{\text{cn}}}
 \end{array} \tag{5.5}$$

□

Lemma 5.3. *There is a homotopy equivalence*

$$\left[\left[\begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array} \right] \right] \sim \left[\mathbf{s}_q \left[\left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right] \right] \xrightarrow{\psi_+} \left[\left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right] \right] \xrightarrow{\chi_-} \left[\right] \left[\right] \right]. \tag{5.6}$$

Proof. Consider the isomorphism of complexes

$$\begin{array}{ccc}
 \left[\left[\left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right] \right] \right] & \xrightarrow{\begin{pmatrix} \chi_- \otimes \mathbb{1} \\ \mathbb{1} \otimes \chi_- \end{pmatrix}} & \left[\left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right] \right] \oplus \left[\left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right] \right] \xrightarrow{(\chi_-, -\chi_-)} \left[\right] \left[\right] \\
 \downarrow \cong & & \downarrow \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \\
 \left[\left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right] \right] \oplus \mathbf{s}_q \left[\left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right] \right] & \xrightarrow{\begin{pmatrix} 0 & \psi_+ \\ \mathbb{1} & 0 \end{pmatrix}} & \left[\left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right] \right] \oplus \left[\left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right] \right] \xrightarrow{(\chi_- \ 0)} \left[\right] \left[\right]
 \end{array} \tag{5.7}$$

The complex at the top line represents $\left[\left[\begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array} \right] \right]$. The complex at the bottom line is a sum of the contractible complex $\left[\left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right] \right] \xrightarrow{\mathbb{1}} \left[\left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right] \right]$ and the complex in the *r.h.s.* of eq.(5.6). □

Adding an extra strand to the homotopy equivalence (5.6) turns it into

$$\left[\left[\begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array} \right] \right] \sim \left[\mathbf{s}_q \left[\left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right] \right] \xrightarrow{\psi_+} \left[\left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right] \right] \xrightarrow{\chi_-} \left[\right] \left[\right] \right]. \tag{5.8}$$

A substitution of the *r.h.s.* complex for the front face of the cube complex (5.3) turns the latter into the complex

$$\left[\begin{array}{c} \text{Diagram} \\ \sim \left[\begin{array}{c} \text{Diagram} \\ \text{Diagram} \end{array} \right] \end{array} \right] \quad (5.9)$$

$$\sim \left[\begin{array}{c} A \xrightarrow{\tilde{\kappa}} B \end{array} \right]. \quad (5.10)$$

Here ξ is a composition of morphisms

$$\left[\begin{array}{c} \text{Diagram} \end{array} \right] \xrightarrow{\mathbb{1} \otimes \chi_- \otimes \mathbb{1}} \left[\begin{array}{c} \text{Diagram} \end{array} \right] \xrightarrow{p_\circ} \text{s} \left[\begin{array}{c} \text{Diagram} \end{array} \right], \quad (5.11)$$

while

$$A = \left[\begin{array}{c} \left[\begin{array}{c} \text{Diagram} \end{array} \right] \xrightarrow{\xi} \text{s} \left[\begin{array}{c} \text{Diagram} \end{array} \right] \end{array} \right], \quad B = \left[\begin{array}{c} \text{Diagram} \\ \text{Diagram} \end{array} \right] \quad (5.12)$$

and the morphism $\tilde{\kappa}$ in the cone (5.10) is determined by the diagram (5.9).

It is well-known (see *e.g.* Lemma 4.7 in [KR] and references therein) that *if we ignore filtration*, then the bimodule appearing at the head of the complex (5.11) splits:

$$\left[\begin{array}{c} \text{Diagram} \end{array} \right] \cong \left[\begin{array}{c} \text{Diagram} \end{array} \right] \oplus \left[\begin{array}{c} \text{Diagram} \end{array} \right], \quad (5.13)$$

where

$$\left[\begin{array}{c} \text{Diagram} \end{array} \right] = \mathbb{Q}[\mathbf{x}, \mathbf{y}] / (p_1(\mathbf{y}) - p_1(\mathbf{x}), p_2(\mathbf{y}) - p_2(\mathbf{x}), p_3(\mathbf{y}) - p_3(\mathbf{x}))$$

and $p_k(\mathbf{x}) = x_1^k + x_2^k + x_3^k$. Moreover,

$$\left[\begin{array}{c} \text{Diagram} \end{array} \right] \cong \ker \xi \quad (5.14)$$

and ξ acts as an isomorphism on the second component of the sum (5.13). In terms of the derived category, the isomorphism (5.14) says that there is a quasi-isomorphism

$$\left[\begin{array}{c} \text{Diagram} \end{array} \right] \simeq \text{Cone}(\xi)$$

and the splitting (5.13) says that one of the canonical maps in the corresponding distinguished triangles is trivial:

$$\left[\left[\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right] \right] \xrightarrow{0} \text{Cone}(\xi) \cong \left[\left[\begin{array}{c} \diagup \diagdown \\ \bullet \end{array} \right] \right] \longrightarrow \left[\left[\begin{array}{c} \bullet \diagup \diagdown \\ \bullet \end{array} \right] \right] \xrightarrow{\xi} \left[\left[\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right] \right]. \quad (5.15)$$

Now consider the canonical resolution of the bimodule (5.13):

$$\begin{aligned} \left[\left[\begin{array}{c} \bullet \diagup \diagdown \\ \bullet \end{array} \right] \right]^{\text{cn}} &= \boxed{\begin{array}{c} \begin{array}{ccccc} & & \mathbf{s} \left[\left[\begin{array}{c} \diagup \diagdown \end{array} \right] \right]^{\text{cn}} & \longrightarrow & \mathbf{s}^2 \left[\left[\begin{array}{c} \diagup \diagdown \end{array} \right] \right]^{\text{cn}} \\ & \nearrow & & \searrow & \\ \left[\left[\begin{array}{c} | \end{array} \right] \right]^{\text{cn}} & \longrightarrow & \mathbf{s} \left[\left[\begin{array}{c} \diagup \diagdown \end{array} \right] \right]^{\text{cn}} & \xrightarrow{\text{Ct}} & \mathbf{s}^2 \left[\left[\begin{array}{c} \diagup \diagdown \end{array} \right] \right]^{\text{cn}} \\ & \searrow & & \nearrow & \\ & & \mathbf{s} \left[\left[\begin{array}{c} \diagup \diagdown \end{array} \right] \right]^{\text{cn}} & \longrightarrow & \mathbf{s}^2 \left[\left[\begin{array}{c} \diagup \diagdown \end{array} \right] \right]^{\text{cn}} \end{array} \\ & & & & \searrow & \nearrow \\ & & & & \mathbf{s}^3 \left[\left[\begin{array}{c} \diagup \diagdown \end{array} \right] \right]^{\text{cn}} \end{array} \end{array} \quad (5.16) \\ &= \boxed{\left[\left[\begin{array}{c} \bullet \diagup \diagdown \\ \bullet \end{array} \right] \right]^{\text{cn}} \xrightarrow{\mathbb{1} \otimes \phi_{\text{II}} \otimes \mathbb{1}} \mathbf{s} \left[\left[\begin{array}{c} \bullet \diagup \diagdown \\ \bullet \end{array} \right] \right]^{\text{cn}}} \end{aligned}$$

The arrows in the large boxed diagram correspond to virtual saddles similar to (2.23). The last cone comes from the presentation (2.20) for the middle blob, its components correspond to the back and front faces of the cube complex. Decomposing the first component of the cone in accordance with Lemma 5.2, we turn the cube complex (5.16) into the following complex

$$\left[\left[\begin{array}{c} \bullet \diagup \diagdown \\ \bullet \end{array} \right] \right]^{\text{cn}} \cong \boxed{\begin{array}{c} \mathbf{s} \left[\left[\begin{array}{c} \diagup \diagdown \end{array} \right] \right]^{\text{cn}} \longrightarrow \mathbf{s}^2 \left[\left[\begin{array}{c} | \end{array} \right] \right]^{\text{cn}} \\ \searrow \quad \nearrow \\ \left[\left[\begin{array}{c} | \end{array} \right] \right]^{\text{cn}} \longrightarrow \mathbf{s} \left[\left[\begin{array}{c} \diagup \diagdown \end{array} \right] \right]^{\text{cn}} \longrightarrow \mathbf{s}^2 \left[\left[\begin{array}{c} \diagup \diagdown \end{array} \right] \right]^{\text{cn}} \\ \searrow \quad \nearrow \\ \mathbf{s} \left[\left[\begin{array}{c} \diagup \diagdown \end{array} \right] \right]^{\text{cn}} \longrightarrow \mathbf{s}^2 \left[\left[\begin{array}{c} \diagup \diagdown \end{array} \right] \right]^{\text{cn}} \longrightarrow \mathbf{s}^3 \left[\left[\begin{array}{c} \diagup \diagdown \end{array} \right] \right]^{\text{cn}} \end{array} \quad (5.17)$$

$$= \boxed{\mathbf{s} \left[\left[\begin{array}{c} \diagup \diagdown \end{array} \right] \right]^{\text{cn}} \xrightarrow{f_{\times, \text{I}}} C_*} \quad (5.18)$$

The first component of the cone (5.18) is the complex in the first row in the big diagram, while the second component is the complex in the other two bottom rows. Since the transformation of the diagram (5.16) into the diagram (5.17) is performed by the splitting (5.4), which is a part of the morphism ξ in the diagram (5.11), it follows that ξ is one of the canonical

morphisms in the distinguished triangle of the cone (5.18):

$$\mathbf{s} \left[\left[\begin{array}{c} \diagup \quad \diagdown \\ \bullet \end{array} \right] \right] \xrightarrow{f_{\times i}} C_* \xrightarrow{\eta} \left[\left[\begin{array}{c} \diagup \quad \bullet \quad \diagdown \\ \bullet \end{array} \right] \right] \xrightarrow{\xi} \mathbf{s} \left[\left[\begin{array}{c} \diagup \quad \diagdown \\ \bullet \end{array} \right] \right]^{\text{cn}}. \quad (5.19)$$

Both distinguished triangles (5.19) and (5.15) are associated with the same morphism ξ , hence they are the same. This identification means that the complex C_* is quasi-isomorphic to the bimodule $\left[\left[\begin{array}{c} \diagup \quad \bullet \quad \diagdown \\ \bullet \end{array} \right] \right]$ and it represents a free resolution of the latter. Hence the filtration of C_* determines the filtration of the bimodule $\left[\left[\begin{array}{c} \diagup \quad \bullet \quad \diagdown \\ \bullet \end{array} \right] \right]$ and we declare the former to be the canonical filtered resolution of the latter:

$$\left[\left[\begin{array}{c} \diagup \quad \bullet \quad \diagdown \\ \bullet \end{array} \right] \right]^{\text{cn}} = \boxed{\begin{array}{ccccc} & & \mathbf{s} \left[\left[\begin{array}{c} \diagup \quad \diagdown \\ \bullet \end{array} \right] \right]^{\text{cn}} & \longrightarrow & \mathbf{s}^2 \left[\left[\begin{array}{c} \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \end{array} \right] \right]^{\text{cn}} & \longrightarrow & \mathbf{s}^3 \left[\left[\begin{array}{c} \diagup \quad \diagdown \quad \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \end{array} \right] \right]^{\text{cn}} \\ \left[\left[\begin{array}{c} | \quad | \quad | \end{array} \right] \right]^{\text{cn}} & \searrow & & \nearrow & & & \\ & \mathbf{s} \left[\left[\begin{array}{c} \diagup \quad \diagdown \end{array} \right] \right]^{\text{cn}} & \longrightarrow & \mathbf{s}^2 \left[\left[\begin{array}{c} \diagup \quad \diagdown \end{array} \right] \right]^{\text{cn}} & \longrightarrow & & \end{array}} \quad (5.20)$$

Thus the splitting (5.13) can be presented as a pair of (unfiltered) isomorphisms

$$\left[\left[\begin{array}{c} \diagup \quad \bullet \quad \diagdown \\ \bullet \end{array} \right] \right] \xrightleftharpoons[\left(\eta \quad \tilde{\xi}\right)]{\left(\tilde{\eta} \quad \xi\right)} \left(\left[\left[\begin{array}{c} \diagup \quad \bullet \quad \diagdown \\ \bullet \end{array} \right] \right] \oplus \mathbf{s} \left[\left[\begin{array}{c} \diagup \quad \diagdown \\ \bullet \end{array} \right] \right] \right),$$

where the untilded homomorphisms ξ and η are filtered and appear in the distinguished triangle (5.19), whereas the tilded homomorphisms $\tilde{\xi}$ and $\tilde{\eta}$ violate filtration. However, it is easy to see that the depth of filtration limits these violation to two units.

Lemma 5.4. *There is a homotopy equivalence (retraction) from A of (5.12) to $\left[\left[\begin{array}{c} \diagup \quad \bullet \quad \diagdown \\ \bullet \end{array} \right] \right]$*

$$h_A \hookrightarrow A = \left[\left[\left[\begin{array}{c} \diagup \quad \bullet \quad \diagdown \\ \bullet \end{array} \right] \right] \xrightarrow{\xi} \mathbf{s} \left[\left[\begin{array}{c} \diagup \quad \diagdown \\ \bullet \end{array} \right] \right] \right] \xrightleftharpoons[f_{*A}]{f_{A*}} \left[\left[\begin{array}{c} \diagup \quad \bullet \quad \diagdown \\ \bullet \end{array} \right] \right], \quad (5.21)$$

such that f_{*A} is filtered, while f_{A*} and h_A violate filtration by two units, and there is a relation $h_A f_{*A} = 0$.

Proof. The homotopy equivalences f_{*A} and f_{A*} are

$$\begin{array}{c} \left[\left[\begin{array}{c} \diagup \quad \bullet \quad \diagdown \\ \bullet \end{array} \right] \right] \\ \downarrow \eta \\ \left[\left[\left[\begin{array}{c} \diagup \quad \bullet \quad \diagdown \\ \bullet \end{array} \right] \right] \xrightarrow{\xi} \mathbf{s} \left[\left[\begin{array}{c} \diagup \quad \diagdown \\ \bullet \end{array} \right] \right] \right] \\ \downarrow \tilde{\eta} \\ \left[\left[\begin{array}{c} \diagup \quad \bullet \quad \diagdown \\ \bullet \end{array} \right] \right] \end{array}$$

(note that $\xi\eta = 0$) while the homotopy h_A is

$$\begin{array}{ccc} \left[\left[\begin{array}{c} \text{diagram} \end{array} \right] \right] & \xrightarrow{\xi} & \mathbf{s} \left[\begin{array}{c} \text{diagram} \end{array} \right] \\ & \nwarrow \tilde{\xi} & \\ \left[\left[\begin{array}{c} \text{diagram} \end{array} \right] \right] & \xrightarrow{\xi} & \mathbf{s} \left[\begin{array}{c} \text{diagram} \end{array} \right] \end{array}.$$

□

Define the complex C_{sym} of eq.(5.2) as

$$C_{\text{sym}} = \left[\left[\begin{array}{c} \text{diagram} \end{array} \right] \right] \xrightarrow{\kappa} B, \quad (5.22)$$

where B is defined by eq.(5.12), while the morphism κ is a composition of f_{A*} and $\tilde{\kappa}$:

$$\begin{array}{ccccc} & & \kappa & & \\ & \searrow & \text{arc} & \swarrow & \\ \left[\begin{array}{c} \text{diagram} \end{array} \right] & \xrightarrow{f_{A*}} & A = \left[\left[\begin{array}{c} \text{diagram} \end{array} \right] \right] \xrightarrow{\xi} \mathbf{s} \left[\begin{array}{c} \text{diagram} \end{array} \right] & \xrightarrow{\tilde{\kappa}} & B \end{array}$$

Since f_{A*} and $\tilde{\kappa}$ are filtered, κ is also filtered.

Lemma 5.5. *There is a homotopy equivalence (retraction)*

$$C_{\text{sym}} = \left[\left[\begin{array}{c} \text{diagram} \end{array} \right] \right] \xrightarrow{\kappa} B \xrightleftharpoons[g]{f} \left[A \xrightarrow{\tilde{\kappa}} B \right] \hookrightarrow h$$

such that f is filtered while g and h violate filtration by at most two units.

Proof. Since there is a relation $h_A f_{*A} = 0$, then in accordance with Lemma (3.4) one can choose homotopy equivalences f and g as

$$\begin{array}{ccc} \left[A \xrightarrow{\tilde{\kappa}} B \right] & & \\ \downarrow f_{A*} \quad \searrow \tilde{\kappa} h_A & & \downarrow \mathbb{1} \\ \left[\left[\begin{array}{c} \text{diagram} \end{array} \right] \right] \xrightarrow{\kappa} B & & \\ \downarrow f_{*A} & & \downarrow \mathbb{1} \\ \left[A \xrightarrow{\tilde{\kappa}} B \right] & & \end{array},$$

while the homotopy h can be chosen as

$$\begin{array}{ccc} \left[A \xrightarrow{\tilde{\kappa}} B \right] & & \\ \downarrow h_A & & \\ \left[A \xrightarrow{\tilde{\kappa}} B \right] & & \end{array}$$

Since f_{*A} and $\tilde{\kappa}$ are filtered, while h_A violates filtration by at most two units, it follows that f is filtered, while g and h violate filtration by at most two units. \square

Combining this lemma with the filtered homotopy equivalence (5.10), we get the following

Corollary 5.6. *There is a homotopy equivalence (retraction)*

$$C_{\text{sym}} \begin{matrix} \xrightarrow{f_{\text{sr}}} \\ \xleftarrow{f_{\text{rs}}} \end{matrix} \left[\begin{array}{c} \text{diagram} \end{array} \right]$$

such that f_{sr} is filtered, while f_{rs} violates filtration by two units.

It remains to show that the complex C_{sym} of eq. (5.22) is symmetric with respect to reflection of the diagrams across the vertical axis. Indeed, the complex has the form

$$C_{\text{sym}} = \left[\begin{array}{c} \left[\begin{array}{c} \text{diagram} \end{array} \right] \xrightarrow{\zeta_u} \left[\begin{array}{c} \text{diagram} \end{array} \right] \longrightarrow \left[\begin{array}{c} \text{diagram} \end{array} \right] \longrightarrow \left[\begin{array}{c} \text{diagram} \end{array} \right] \\ \xrightarrow{\zeta_d} \left[\begin{array}{c} \text{diagram} \end{array} \right] \longrightarrow \left[\begin{array}{c} \text{diagram} \end{array} \right] \longrightarrow \left[\begin{array}{c} \text{diagram} \end{array} \right] \end{array} \right]$$

where the unmarked arrows denote the morphisms χ_- similar to (2.24), while ζ_u and ζ_d are components of the morphism κ . Since the homotopy equivalence from the complex (5.3) to the complex (5.9) involves the homotopy of only the front face of the cube (5.3), the morphisms ζ_u are compositions of η and virtual saddle morphisms:

$$\begin{array}{c} \left[\begin{array}{c} \text{diagram} \end{array} \right] \xrightarrow{\eta} \left[\begin{array}{c} \text{diagram} \end{array} \right] \\ \nearrow \zeta_u \quad \searrow \zeta_d \\ \left[\begin{array}{c} \text{diagram} \end{array} \right] \quad \left[\begin{array}{c} \text{diagram} \end{array} \right] \end{array}$$

Now it is easy to verify that ζ_u and ζ_d are canonical morphisms associated with two presentations of $\left[\begin{array}{c} \text{diagram} \end{array} \right]$ as cones originating from the splits of the canonical complex (5.20):

$$\left[\begin{array}{c} \text{diagram} \end{array} \right] \simeq \left[\begin{array}{c} \left[\begin{array}{c} \text{diagram} \end{array} \right] \longrightarrow \mathbf{s}^2 \left[\begin{array}{c} \text{diagram} \end{array} \right] \end{array} \right] \simeq \left[\begin{array}{c} \left[\begin{array}{c} \text{diagram} \end{array} \right] \longrightarrow \mathbf{s}^2 \left[\begin{array}{c} \text{diagram} \end{array} \right] \end{array} \right]$$

the first (resp. the second) cone corresponding to splitting off the upper-right (resp. upper-left) side of the hexagonal complex (5.20). Now the symmetry of the complex C_{sym} becomes apparent, and this completes the proof of Theorem 5.1. \square

6. MARKOV MOVES

6.1. Filtered homology of a link diagram. Let L_β be the oriented unframed link diagram constructed by the circular closure of an n -strand braid word β . We define the filtered complex $\llbracket L_\beta \rrbracket$ of a link diagram L_β as the result of replacing the filtered $\mathbb{Q}[\mathbf{x}, \mathbf{y}]$ -modules in the complex ${}_y\llbracket \beta \rrbracket_x$ by their derived tensor products with the diagonal bimodule corresponding to the n -strand identity braid $\mathbb{1}_n$ with a special grading shift:

$$\llbracket L_\beta \rrbracket = \mathbf{t}_s^n \left({}_y\llbracket \beta \rrbracket_x \overset{\mathbf{L}}{\otimes}_{\mathbb{Q}[\mathbf{x}, \mathbf{y}]} {}_y\llbracket \mathbb{1}_n \rrbracket_x \right), \quad (6.1)$$

where ${}_y\llbracket \mathbb{1}_n \rrbracket_x = \mathbb{Q}[\mathbf{x}, \mathbf{y}]/(y_1 - x_1, \dots, y_n - x_n)$. A derived tensor product of two filtered complexes in the derived category $\mathbf{D}(\mathbb{Q}[\mathbf{x}, \mathbf{y}] - \mathbf{fi}, \mathbf{g})$ is an object in the homotopy category of filtered complexes $\mathbf{K}(\mathbb{Q} - \mathbf{fi}, \mathbf{g})$. Hence $\llbracket L_\beta \rrbracket$ is an object of the category $\mathbf{Ch}(\mathbf{K}(\mathbb{Q} - \mathbf{fi}, \mathbf{g}))$ of chain complexes over $\mathbf{K}(\mathbb{Q} - \mathbf{fi}, \mathbf{g})$. Note that generally a complex in the category $\mathbf{K}(\mathbb{Q} - \mathbf{fi}, \mathbf{g})$ is not homotopy equivalent to its homology because of the filtration.

Remark 6.1. *Two braid words β and β' yield the same link diagram by circular closure if they are related by the first Markov move, that is, if there exist braid words β_1 and β_2 such that $\beta = \beta_2\beta_1$, $\beta' = \beta_1\beta_2$. Obviously, the complex $\llbracket L_\beta \rrbracket$ is invariant under the first Markov move, because of the isomorphisms*

$$\llbracket L_\beta \rrbracket \cong \mathbf{t}_s^n \left({}_y\llbracket \beta_1 \rrbracket_x \overset{\mathbf{L}}{\otimes}_{\mathbb{Q}[\mathbf{x}, \mathbf{y}]} {}_x\llbracket \beta_2 \rrbracket_y \right) \cong \llbracket L_{\beta'} \rrbracket. \quad (6.2)$$

Hence the complex (6.1) is indeed determined by the link diagram L_β rather than by the braid word β .

An application of the homology with respect to the inner differential (that is, of the category $\mathbf{K}(\mathbb{Q} - \mathbf{fi}, \mathbf{g})$) and with respect to the outer differential (that is, of category $\mathbf{Ch}(-)$) produces two more invariants of the diagram L_β :

$$\mathbf{H}_{\text{in}}(\llbracket L_\beta \rrbracket) = \mathbf{t}_s^n \mathbf{HH}(\llbracket \beta \rrbracket), \quad (6.3)$$

(HH being the Hochschild homology), which is an object of the category $\mathbf{Ch}(\mathbb{Q} - \mathbf{fi}, \mathbf{g})$ of filtered, graded chain complexes, and the filtered triply graded homology

$$\mathbf{H}^{(3)}(L_\beta) = \mathbf{H}_{\text{out}}(\mathbf{H}_{\text{in}}(\llbracket L_\beta \rrbracket)).$$

Remark 6.2. *Since $\mathbf{H}_{\text{in}}(\llbracket L_\beta \rrbracket)$ is derived from $\llbracket L_\beta \rrbracket$ and $\mathbf{H}^{(3)}(L_\beta)$ is derived from $\mathbf{H}_{\text{in}}(\llbracket L_\beta \rrbracket)$, then for two braid words β and β' there is a sequence of implications:*

$$\llbracket L_{\beta'} \rrbracket \sim \llbracket L_\beta \rrbracket \quad \Rightarrow \quad \mathbf{H}_{\text{in}}(\llbracket L_{\beta'} \rrbracket) \sim \mathbf{H}_{\text{in}}(\llbracket L_\beta \rrbracket) \quad \Rightarrow \quad \mathbf{H}^{(3)}(\llbracket L_{\beta'} \rrbracket) \cong \mathbf{H}^{(3)}(\llbracket L_\beta \rrbracket).$$

6.2. Markov moves. Fix an n -strand braid word β . By definition, the braids

$$\beta' = (\sigma_1 \sqcup \mathbb{1}_n)(\mathbb{1}_1 \sqcup \beta), \quad \beta'' = (\sigma_1^{-1} \sqcup \mathbb{1}_n)(\mathbb{1}_1 \sqcup \beta) \quad (6.4)$$

are the result of applying Markov moves 2A and 2B to β , so the circular closures L_β , $L_{\beta'}$ and $L_{\beta''}$ represent isotopic link diagrams.

Theorem 6.3. *For any braid word β and for the braid words (6.4), which result from applying Markov moves 2A and 2B to β , there are homotopy equivalences*

$$[L_{\beta'}] \sim [L_\beta], \quad H_{\text{in}}([L_{\beta''}]) \sim H_{\text{in}}([L_\beta]). \quad (6.5)$$

Remark 6.4. *In view of Remark 6.2, the first equivalence of (6.5) is stronger than the second equivalence.*

Proof of Theorem 6.3. Denote by \widetilde{L}_β the diagram of a (1,1)-tangle resulting from the circular closure of all strands of β except the first strand. Define the corresponding complex $[\widetilde{L}_\beta]$ similar to eq.(6.1):

$${}_{y_1}[\widetilde{L}_\beta]_{x_1} = \mathbf{t}_s^{n-1} \left({}_{\mathbf{y}}[\beta]_{\mathbf{x}} \overset{\text{L}}{\otimes}_{\mathbb{Q}[\mathbf{x}', \mathbf{y}']} {}_{\mathbf{y}'}[\mathbb{1}_{n-1}]_{\mathbf{x}'} \right),$$

where $\mathbf{x}' = x_2, \dots, x_n$ and \mathbf{y}' is defined similarly. Also for any 2-strand diagram \bowtie we define the complex of its partial (1-strand) closure as a derived tensor product similar to (6.2):

$${}_{y_2}[\bowtie]_{x_2} = \mathbf{t}_s \left({}_{\mathbf{x}}[\bowtie]_{\mathbf{y}} \overset{\text{L}}{\otimes}_{\mathbb{Q}[x_1, y_1]} {}_{y_1}[\mathbb{1}]_{x_1} \right). \quad (6.6)$$

Now it is easy to see that the complexes of the braid words (6.4) can be presented as derived tensor products

$$[L_{\beta'}] \cong \mathbf{t}_s \left({}_{x_2}[\bowtie]_{y_2} \overset{\text{L}}{\otimes}_{\mathbb{Q}[x_2, y_2]} {}_{y_2}[\widetilde{L}_\beta]_{x_2} \right), \quad (6.7)$$

$$[L_{\beta''}] \cong \mathbf{t}_s \left({}_{x_2}[\bowtie]_{y_2} \overset{\text{L}}{\otimes}_{\mathbb{Q}[x_2, y_2]} {}_{y_2}[\widetilde{L}_\beta]_{x_2} \right), \quad (6.8)$$

whereas

$$[L_\beta] \cong \mathbf{t}_s \left({}_{y_1}[\mathbb{1}]_{x_1} \overset{\text{L}}{\otimes}_{\mathbb{Q}[x_1, y_1]} {}_{y_1}[\widetilde{L}_\beta]_{x_1} \right). \quad (6.9)$$

Lemma 6.5. *There are homotopy equivalences*

$$[\bowtie] \sim [\mathbb{1}], \quad [\bowtie] \sim \left[\mathbf{t} \begin{array}{c} \boxed{\begin{array}{c} \mathbf{a}[\mathbb{1}] \\ \downarrow \mathbb{1} \\ \mathbf{s}[\mathbb{1}] \end{array}} \xrightarrow{\mathbb{1}} [\mathbb{1}] \end{array} \right]. \quad (6.10)$$

Replacing the first factor in the tensor product (6.7) with the help of the first homotopy equivalence of (6.10) converts the *r.h.s.* of (6.7) into the *r.h.s.* of (6.9), thus proving the first homotopy equivalence of (6.5).

A substitution of the second homotopy equivalence of (6.10) into eq.(6.8) yields the presentation of $\llbracket L_{\beta''} \rrbracket$ as a cone of two tensor products:

$$\llbracket L_{\beta''} \rrbracket \sim \mathbf{t}_s \left[\mathbf{t} \left(A \overset{\mathbf{L}}{\otimes}_{\mathbb{Q}[x_2, y_2]} {}_{y_2} \llbracket \widetilde{L}_{\beta} \rrbracket_{x_2} \right) \xrightarrow{f \otimes \mathbf{1}} \left({}_{y_2} \llbracket \rrbracket_{x_2} \overset{\mathbf{L}}{\otimes}_{\mathbb{Q}[x_2, y_2]} {}_{y_2} \llbracket \widetilde{L}_{\beta} \rrbracket_{x_2} \right) \right] \quad (6.11)$$

where

$$A = \boxed{\left(\mathbf{a}_{{}_{y_2} \llbracket \rrbracket_{x_2}} \xrightarrow{\mathbf{1}} \left(\mathbf{s}_{{}_{y_2} \llbracket \rrbracket_{x_2}} \right) \right)}.$$

If we forget filtration, then the cone A is contractible, hence

$$\mathrm{H}_{\mathrm{in}} \left(A \overset{\mathbf{L}}{\otimes}_{\mathbb{Q}[x_2, y_2]} {}_{y_2} \llbracket \widetilde{L}_{\beta} \rrbracket_{x_2} \right) = 0$$

and the application of H_{in} to both sides of the homotopy equivalence (6.11) produces the second homotopy equivalence of (6.5). \square

Proof of Lemma 6.5. We compute the derived tensor product in (6.6) for the *r.h.s.* of this presentation by using the canonical filtered resolution for 2-strand graphs and presenting ${}_{y_1} \llbracket \rrbracket_{x_1}$ as the diagonal bimodule ${}_{y_1} M_{x_1}^{\Delta} = \mathbb{Q}[x_1, y_1]/(y_1 - x_1)$:

$$\left[\text{diagram} \right]_{x_2} \cong \mathbf{t}_s \left({}_{\mathbf{y}} \llbracket \text{diagram} \rrbracket_{\mathbf{x}}^{\mathrm{cn}} \otimes_{\mathbb{Q}[x_1, y_1]} {}_{y_1} M_{x_1}^{\Delta} \right).$$

Taking the tensor product with ${}_{y_1} M_{x_1}^{\Delta}$ amounts to setting $x_1 = y_1$ and forgetting the structure of a module over $\mathbb{Q}[x_1, y_1]$. After we replace the dummy variables $x_1 = y_1$ with another dummy variable $z = (x_2 + y_2) - (x_1 + y_1)$, the complexes of 1-strand closures of the diagrams $\rangle \langle$ and \times take the form (*cf.* eq.(2.8))

$$\left[\text{diagram} \right]_{x_2} \simeq \mathbf{t}_s \left({}_{y_2} \Delta_{x_2} \otimes \boxed{\mathbf{a}_q \mathbb{Q}[z] \xrightarrow{0} \mathbb{Q}[z]} \right) \simeq {}_{y_2} \llbracket \rrbracket_{x_2} \otimes \mathbf{t}_s (\mathbf{a}_q \mathbb{Q}[z] \oplus \mathbb{Q}[z]) \quad (6.12)$$

$$\left[\text{diagram} \right]_{x_2} \simeq \mathbf{t}_s \left({}_{y_2} \Delta_{x_2} \otimes \boxed{\mathbf{a}_q \mathbb{Q}[z] \xrightarrow{z} \mathbb{Q}[z]} \right) \simeq \mathbf{t}_s {}_{y_2} \llbracket \rrbracket_{x_2} \quad (6.13)$$

In the last line we used a splitting $\mathbb{Q}[z] \cong \mathbb{Q}[z]/(z) \oplus z\mathbb{Q}[z]$, which isolates a contractible subcomplex:

$$\boxed{\mathbf{a}_q \mathbb{Q}[z] \xrightarrow{z} \mathbb{Q}[z]} \cong \mathbb{Q}[z]/(z) \oplus \boxed{\mathbf{a}_q \mathbb{Q}[z] \xrightarrow{\mathbf{1}} z\mathbb{Q}[z]} \sim \mathbb{Q}[z]/(z) \cong \mathbb{Q}.$$

Now we apply the 1-strand closures to the definitions (2.19) and (2.20). We begin with the complex (2.19). After the closure, the morphism ϕ_\times becomes homotopically trivial, hence

$$\begin{aligned} \left[\left[\text{diagram} \right] \right] &\cong \mathbf{t}_s \mathbf{a}_q \left[\begin{array}{c} \left[\text{diagram} \right] \xrightarrow{\mathbb{1}} \left[\text{diagram} \right] \\ \downarrow 0 \\ \mathbf{t}_a^{-1} \left[\text{diagram} \right] \end{array} \right] \\ &\cong \mathbf{t}_s^{-1} \left[\text{diagram} \right] \oplus \mathbf{t}_s \mathbf{a}_q \left[\left[\text{diagram} \right] \xrightarrow{\mathbb{1}} \mathbf{t}_a^{-1} \left[\text{diagram} \right] \right] \sim \left[\text{diagram} \right] \end{aligned}$$

The last homotopy equivalence is due to the contractibility of the complex in square brackets, and we also used the quasi-isomorphism (6.13). Thus we proved the first relation of (6.10).

Now we apply closure to the complex (2.20):

$$y_2 \left[\text{diagram} \right]_{x_2} \cong y_2 \left[\text{diagram} \right]_{x_2} \otimes B,$$

where

$$\begin{aligned} B &\cong (\mathbf{t}_s \mathbf{a}_q)^{-1} \left[\mathbf{t}_s \mathbf{t}_a \left[\begin{array}{c} (\mathbf{a}_q \mathbb{Q}[z] \oplus \mathbb{Q}[z]) \xrightarrow{\mathbb{1}} \mathbf{t}_s (\mathbf{a}_q \mathbb{Q}[z] \oplus \mathbb{Q}[z]) \\ \downarrow (1 \ 0) \\ \mathbf{s}_q (\mathbb{Q}[z]/(z)) \end{array} \right] \right] \\ &\cong \left[\mathbf{t} \left[\begin{array}{c} \mathbf{a} \mathbb{Q}[z]/(z) \xrightarrow{\mathbb{1}} \mathbb{Q}[z]/(z) \\ \downarrow \mathbb{1} \\ \mathbf{s} \mathbb{Q}[z]/(z) \end{array} \right] \right] \\ &\quad \oplus \left[\mathbf{t}_a (z \mathbb{Q}[z] \oplus \mathbf{a}_q^{-1} \mathbb{Q}[z]) \xrightarrow{\mathbb{1}} (z \mathbb{Q}[z] \oplus \mathbf{a}_q^{-1} \mathbb{Q}[z]) \right] \end{aligned} \quad (6.14)$$

Since the complex in the last line is contractible, the second homotopy equivalence of (6.10) follows. \square

7. FILTERED HOMOLOGY OF TWO-STRAND TORUS BRAIDS AND LINKS

Denote $\sigma^{-1} = \text{diagram}$.

Theorem 7.1. *For $n \geq 0$, the complex of the braid word $\sigma^{-n} = (\sigma^{-1})^n$ has a form*

$$[\sigma^{-2n}] \sim (\mathbf{t}_s \mathbf{a}_q)^{-2n} \mathbf{t}_a \left[\underbrace{\mathbf{t}_{\text{tot}}^{2n-1} [\text{diagram}] \xrightarrow{\psi_+} \dots \xrightarrow{\psi_+} \mathbf{t}_{\text{tot}}^2 [\text{diagram}] \xrightarrow{\psi_-} \mathbf{t}_{\text{tot}} [\text{diagram}] \xrightarrow{\psi_+} [\text{diagram}]}_{2n} \xrightarrow{\chi_-} \mathbf{t}_a^{-1} [\text{diagram}] \right] \quad (7.1)$$

$$[\sigma^{-(2n+1)}] \sim (\mathbf{t}_s \mathbf{a}_q)^{-(2n+1)} \mathbf{t}_a \left[\underbrace{\mathbf{t}_{\text{tot}}^{2n} [\text{diagram}] \xrightarrow{\psi_-} \dots \xrightarrow{\psi_-} \mathbf{t}_{\text{tot}} [\text{diagram}] \xrightarrow{\psi_+} [\text{diagram}]}_{2n+1} \xrightarrow{\chi_-} \mathbf{t}_a^{-1} [\text{diagram}] \right] \quad (7.2)$$

where $\mathbf{t}_{\text{tot}} = \mathbf{t}_a \mathbf{s}_q$.

Proof. This theorem is proved by induction over n . It obviously holds for $n = 0$ and for $n = 1$. The step of induction is proved with the help of the following lemma whose proof we leave for the readers. \square

Lemma 7.2. *There are homotopy equivalences*

$$\left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \end{array} \right] \sim \mathbf{t}_s \left[\begin{array}{c} \diagup \quad \diagdown \end{array} \right], \quad \left[\begin{array}{c} \diagup \quad \diagdown \\ \bullet \end{array} \right] \sim \mathbf{t}_s \left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \end{array} \right]. \quad (7.3)$$

which make the following diagram commutative:

$$\begin{array}{ccccccc} \left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \end{array} \right] & \xrightarrow{\mathbb{1} \otimes \psi_-} & \left[\begin{array}{c} \diagup \quad \diagdown \end{array} \right] & \xrightarrow{\mathbb{1} \otimes \psi_+} & \left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \end{array} \right] & \xrightarrow{\chi_- \otimes \mathbb{1}} & \left[\begin{array}{c} \diagup \quad \diagdown \end{array} \right] \\ \downarrow \sim & & \downarrow \sim & & \downarrow \text{sim} & & \downarrow \cong \\ \left[\begin{array}{c} \diagup \quad \diagdown \end{array} \right] & \xrightarrow{\psi_+} & \left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \end{array} \right] & \xrightarrow{\psi_-} & \left[\begin{array}{c} \diagup \quad \diagdown \end{array} \right] & \xrightarrow{\theta} & \left[\begin{array}{c} \diagup \quad \diagdown \end{array} \right] \end{array}$$

where θ is the morphism

$$\left[\begin{array}{c} \diagup \quad \diagdown \end{array} \right] \xrightarrow{\psi_+} \left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \end{array} \right] \xrightarrow{\chi_-} \left[\begin{array}{c} \diagup \quad \diagdown \end{array} \right] \cong \left[\begin{array}{c} \diagup \quad \diagdown \end{array} \right]$$

Theorem 7.3. *The filtered triply graded homology homology of the circular closure of the braid word σ^{-n} is*

$$\mathbf{H}^{(3)}(L_{\sigma^{-(2n+1)}}) \cong (\mathbf{t}_s \mathbf{a}_q)^{-2n} \mathbf{H}^{(3)}(\bigcirc) \otimes \left(\mathbb{Q} \oplus \mathbf{t}_a \bigoplus_{k=0}^{n-1} \mathbf{t}_{\text{tot}}^{2k+1} (\mathbf{s}_q \mathbb{Q} \oplus \mathbf{a}_q^{-1} \mathbb{Q}) \right),$$

$$\mathbf{H}^{(3)}(L_{\sigma^{-2n}}) \cong (\mathbf{t}_s \mathbf{a}_q)^{-1} \mathbf{H}^{(3)}(L_{\sigma^{-2n+1}}) \oplus \mathbf{t}_s^{2n-1} \mathbf{a}_q^{-1} \mathbf{t}_a \mathbf{H}^{(3)}(\bigcirc) \otimes (\mathbf{a}_q \mathbf{s}_q \mathbb{Q}[z] \oplus \mathbf{s}_q \mathbb{Q}[z] \oplus \mathbb{Q}).$$

In the latter formula it is assumed that $n \geq 1$.

This theorem is an easy corollary of the following lemma:

Lemma 7.4. *An application of the Hochschild homology functor HH to the sequence of morphisms*

$$\llbracket \bowtie \bullet \rrbracket \xrightarrow{\psi_-} \llbracket \bowtie \rrbracket \xrightarrow{\psi_+} \llbracket \bullet \bowtie \rrbracket \xrightarrow{\chi_-} \llbracket \rangle \langle \rrbracket \quad (7.4)$$

apart from the common factor $\mathbf{t}_s^{-1}\mathrm{H}^{(3)}(\bigcirc)$, where

$$\mathrm{H}^{(3)}(\bigcirc) = \mathbf{t}_s \left(\mathbf{a}_q \mathbb{Q}[w] \oplus \mathbb{Q}[w] \right)$$

is the homology of the unknot diagram, yields the following sequence of maps between graded, filtered vector spaces

$$\begin{array}{ccc} \mathbf{a}_q \mathbf{q}^2 \mathbb{Q}[z] & \xrightarrow{z} & \mathbf{a}_q \mathbf{s}_q \mathbb{Q}[z] & \mathbf{a}_q \mathbf{q}^2 \mathbb{Q}[z] & \xrightarrow{z} & \mathbf{a}_q \mathbb{Q}[z] \\ \oplus & & \oplus & \oplus & & \oplus \\ \mathbb{Q}[z] & \xrightarrow{1} & \mathbf{s}_q \mathbb{Q}[z] & \mathbb{Q}[z] & \xrightarrow{1} & \mathbb{Q}[z] \\ & & \oplus & & & \\ & & \mathbb{Q} & & & \end{array} \quad (7.5)$$

where z and w are ‘dummy’ variables, and each column in the diagram represents the Hochschild homology of the corresponding module in the sequence (7.4).

Sketch of the proof. After we pass from the variables \mathbf{x} and \mathbf{y} to the variables (2.10), the canonical resolution of the bimodule of every diagram \bowtie appearing in (7.4) splits into the tensor product (over \mathbb{Q}) of the complexes of $\mathbb{Q}[x_+, y_+]$ -modules and complexes of $\mathbb{Q}[x_-, y_-]$ -modules: $\llbracket \bowtie \rrbracket^{\mathrm{cn}} \cong {}_{y_+} \Delta_{x_+} \otimes \llbracket \bowtie \rrbracket^-$ (cf. eq. (2.11)), so that apart from the common factor ${}_{y_+} \Delta_{x_+}$, on which the morphisms (7.4) act as identity, the sequence (7.4) becomes

$$\begin{array}{ccccccc} \llbracket \bullet \bowtie \rrbracket^- & \xrightarrow{\psi_-} & \llbracket \bowtie \rrbracket^- & \xrightarrow{\psi_+} & \llbracket \bullet \bowtie \rrbracket^- & \xrightarrow{\chi_-} & \llbracket \rangle \langle \rrbracket^- \\ \downarrow = & & \downarrow = & & \downarrow = & & \downarrow = \\ \boxed{\begin{array}{c} \llbracket \rangle \langle \rrbracket^- \\ \downarrow \phi_{\parallel} \\ \llbracket \bowtie \rrbracket^- \end{array}} & \xrightarrow{\mathbb{1}} & \boxed{\begin{array}{c} \llbracket \bowtie \rrbracket^- \\ \downarrow \phi_{\times} \\ \llbracket \rangle \langle \rrbracket^- \end{array}} & \xrightarrow{\mathbb{1}} & \boxed{\begin{array}{c} \llbracket \rangle \langle \rrbracket^- \\ \downarrow \phi_{\parallel} \\ \llbracket \bowtie \rrbracket^- \end{array}} & \xrightarrow{\mathbb{1}} & \llbracket \rangle \langle \rrbracket^- \end{array} \quad (7.6)$$

where the $\mathbb{Q}[x_-, y_-]$ -modules $\llbracket \rangle \langle \rrbracket^-$ and $\llbracket \bowtie \rrbracket^-$ are defined by eq. (2.12), while the homomorphisms ϕ_{\parallel} and ϕ_{\times} are defined by eq. (2.13). The application of Hochschild homology to the resolutions amounts to taking the quotient over the relation $x_+ = y_+$, which transforms the common factor ${}_{y_+} \Delta_{x_+}$ into $\mathbf{t}_s^{-1}\mathrm{H}^{(3)}(\bigcirc)$, as well as over the relation $x_- = y_-$, which transforms the chain of bimodules (7.6) into the chain of vector spaces (7.5). \square

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